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TRANSVERSE DISPERSION OF CONTAMINANT IN GROUND WATER

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FINAL REPORT

A. SUMMARY

(1) Research Grant No. WP-00575

Title: Transverse Dispersion of Contaminant in Ground Water

(2) Principal Investigator: Wen-Hsiung Li, Professor of Civi Engineering
Syracuse University, Syracuse, N. Y.

(3) Sponsoring Institutions: Public Health Service, Department of Health, Education and Welfare; and Federal Water Pollution Control Administration, Department of the Interior.

(4) Period covered by report: September 1, 1963 to December 31, 1967.

(5) Date of preparation of report: December 31, 1967.

(6) Summary statement:

The coefficient of transverse dispersion has been determined experimentally with the Reynolds number between 0.3 and 380. It is found that inertial forces are important and the flow pattern of the contaminant varies with the Reynolds number even in a flow obeying Darcy's law. Empirical formulas for the coefficient of dispersion are suggested.

Solutions have been obtained for the dispersion of contaminant from a line source and from a point source in a non-uniform flow of ground water. Variation of soil properties and velocity along the flow modifies the maximum concentration at each cross section, while transverse variation of velocity causes a lateral shift of the trail of contaminant, and destroys the symmetry of concentration distribution at a cross section. The contamination of a well by dispersion is used as an example of application.

The effects of gravity and differences in density and viscosity on the dispersion at the interface of two miscible fluids have been studied. It is found that these effects are usually negligible. This conclusion has been verified experimentally. Solutions have been obtained for dispersion at the interface of two fluids in two-diemnsional flow. These solutions are found to be valid for dispersion at the interface of two fluids in three-dimensional flow.

B. DETAILED REPORT

(1) Description of Research Accomplishments

a. Introduction

When a miscible contaminant is introduced in a ground-water flow, the contaminant will disperse through the interconnected pores of the soil. When the Reynolds number vD/ν is larger than 10^{-3} (in which v is the seepage velocity, ν denotes the kinematic viscosity of the fluid, and D represents the grain size)*, molecular diffusion in the fluid is negligible compared with this mechanical dispersion.[1] While there is dispersion in the direction along the flow as well as transverse to the flow, the transverse dispersion dominates in many engineering problems; e.g., the dispersion of contaminant from a source in a ground-water flow, and the dispersion at the interface of two fluids in a shear flow. It is the purpose of this investigation to study these problems. The progress made in this project research is as follows.

(i) In Sections b and c, experimental data of the coefficient of transverse dispersion are presented and discussed. In these experiments, the Reynolds number varies over the wide range of 0.3 to 380. These data reveal the effect on dispersion of the inertial forces. A comparison of the experimental results and an existing statistical theory of dispersion suggests directions for improvement of the theory.

(ii) In a non-uniform flow, the seepage velocity $\vec{v}$ and the tensor coefficient T of dispersion vary from point to point. The equation describing

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* The letter symbols adopted for use in this report are defined where they first appear and listed in Section ℓ below.

1 Harleman, D. R. F., and Rumer, R. R. Jr., "Longitudinal and Lateral Dispersion in an Isotropic Porous Medium", Journ. Fluid Mech., Part 3, Vol. 16, July 1963

the concentration distribution c of the contaminant (in mass per unit volume of soil) is [2,3]

$$\frac{\partial c}{\partial t} + \nabla \cdot (\vec{v} c) = \nabla \cdot [\rho T \nabla (\frac{c}{\rho})] \qquad (1)$$

where t represents time, and ρ is the density of the fluid. For a non-uniform flow, this equation is one with variable coefficients. Generally, the solution of this equation is difficult[4] and has to be performed numerically. For engineering purposes, it is important to seek adequate and simple solutions so that numerical procedure can be avoided. Such solutions have been obtained for the steady dispersion from a line source and from a point source in a non-uniform flow. These relatively simple solutions are presented in Sections d, g, and h. They are shown to be adequate by comparison with accurate numerical solutions in Section e. Examples of application of these solutions are presented in Section f.

(iii) The dispersion at the interface of two miscible fluids in shear flow is discussed in Section i. In Section j, solutions for the dispersion at the interface of two fluids in two-dimensional flow are presented. While the method of determining the shape of the interface in such a flow is well known[5], these solutions describe the mixing at the interface. In Section k, these solutions are shown to be applicable to three-dimensional interfaces.

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2 Bachmat, Y., and Bear, J. "The General Equations of Hydrodynamic Dispersion in Homogeneous, Isotropic, Porous Media", Journal Geophys. Research, Vol. 69, No. 12, 1964.

3 Bird, R. B., Stewart, W. E., and Lightfoot, E. N., Transport Phenomena, John Wiley and Sons, New York, 1962, p. 502.

4 Hille, E., "Functional Analysis and Semi-Groups", Amer. Math. Soc. Coll., Vol. 31, 1948.

5 Polubarinov-Kochina, P. Ya., Theory of Ground Water Movement, Princeton Univ. Press, 1962, p. 352.

b. Experimental Data on Transverse Dispersion

(The materials in Sections b and c are included in a paper entitled "Experiments on Lateral Dispersion in Porous Media" in the Nov. 1966 issue of Journ. of Hydraulics Div., Proc. of Amer. Soc. Civ. Engrs.)

One of the objectives of this investigation is to determine experimentally, over a wide range of the Reynolds number, the component of the tensor coefficient T for transverse dispersion (to be called the coefficient k of transverse dispersion). Three materials were used in the experiments, and they were sand, pebble, and gravel. Their size-distributions obtained from sieve analysis are shown in Fig. 1. The pebble was the least uniform of the three materials. The 50% sizes of the sand, pebble, and gravel were 0.045, 0.15 and 0.49 inch respectively. The typical shapes of the materials are shown in Fig. 2. The porosities of the beds were 0.35, 0.34, and 0.40 for the sand, pebble, and gravel respectively.

The closed channel used in the experiments was 15 inches wide, and 3 inches deep, and is shown schematically in Fig. 3. The channel was filled with a bed-material between the screens at the inlet chambers and the outlets. The length of the bed was 22 feet for the sand and pebble, and was $7\frac{3}{4}$ feet for the gravel. The ceiling of the channel was lined with a sheet of soft rubber. To one of the inlet chambers was admitted filtered water at a constant rate from a constant-head tank. To the other chamber was supplied, at the same rate with an identical arrangement, water with 7 ppm of uranine dye. The elevation of the constant-head tanks could be varied from test to test. There were 36 evenly spaced $\frac{1}{4}$-inch holes across the width of the exit-end to serve as outlets. The details of one of these outlets is shown in Fig. 4. The discharge from these outlets was collected individually and simultaneously, and the dye concentration of the outflow from each outlet was determined with a calibrated spectrophoto-

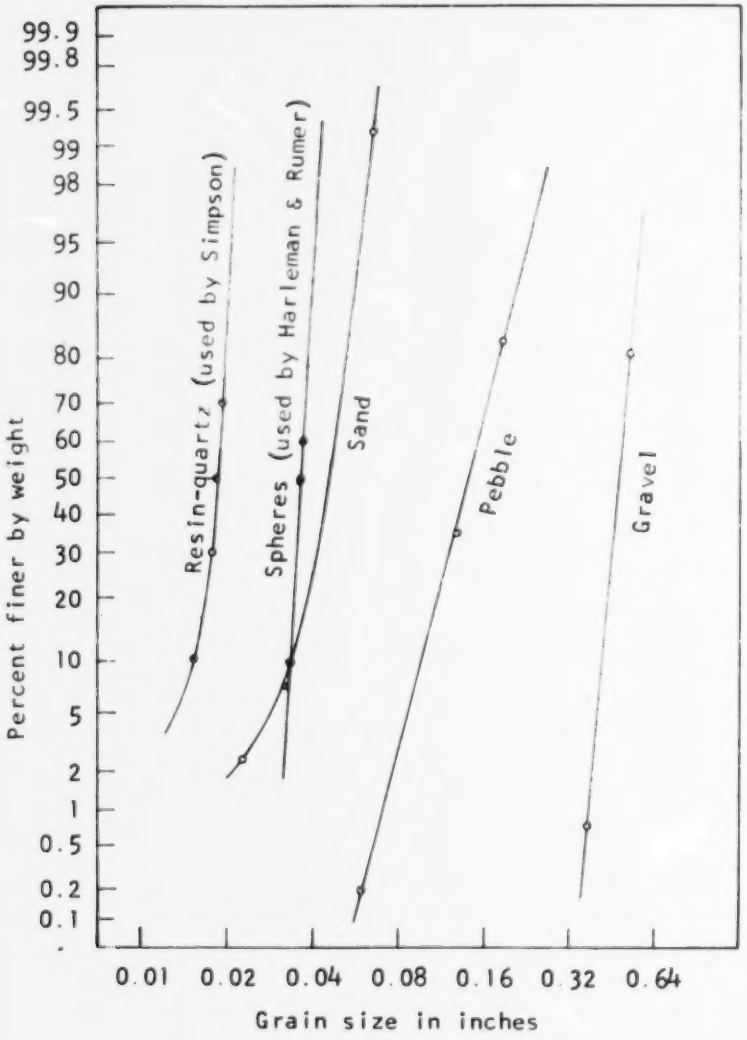

Fig. 1. Size Distribution of Bed Materials

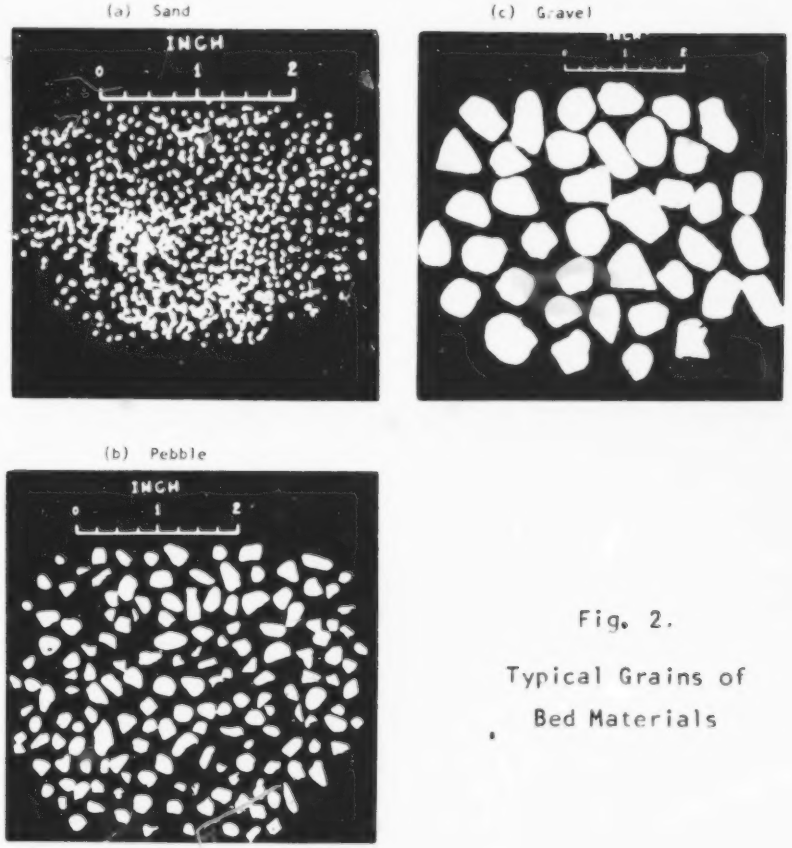

Fig. 2.

Typical Grains of Bed Materials

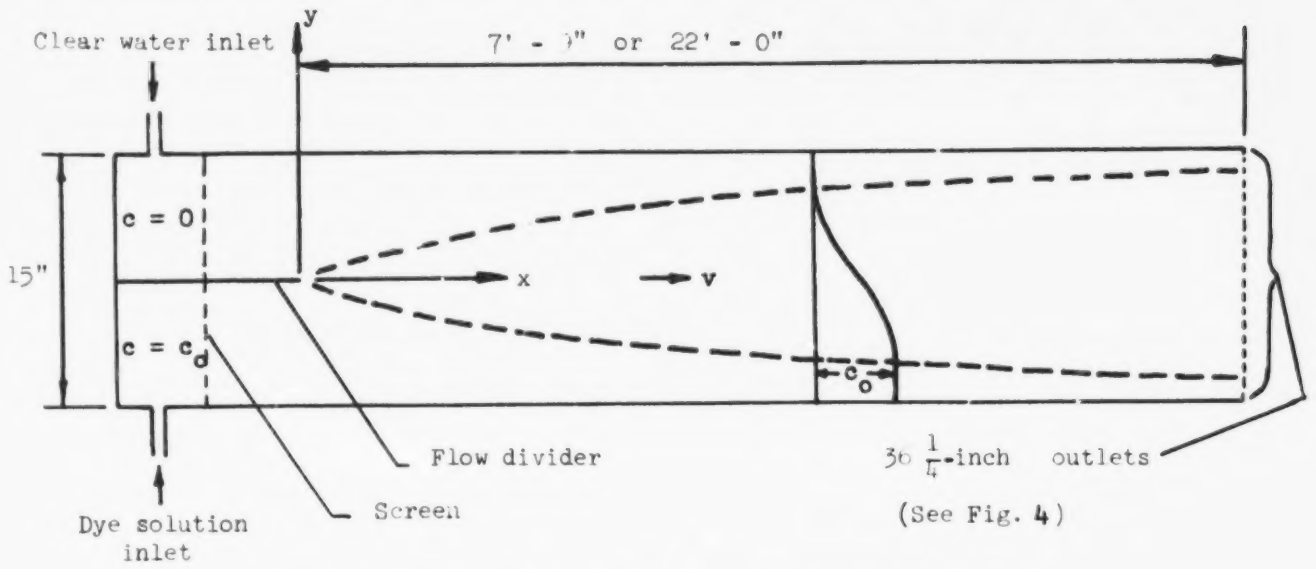

Fig. 3. Porous bed used in experiments.

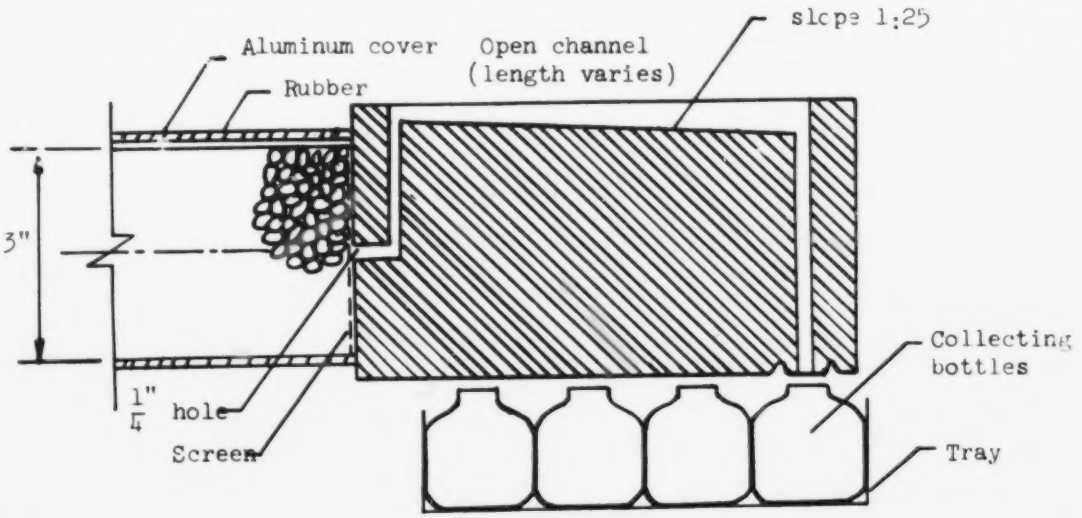

Fig. 4. Details of an outlet.

meter. A typical concentration distribution among the outlets is shown in Fig. 5.

The concentration distribution c(x,y) in Fig. 3 can be obtained from Eq. 1. In this case, longitudinal dispersion is negligible[1] and Eq. 1 becomes

$$v \frac{\partial c}{\partial x} = k \frac{\partial^2 c}{\partial y^2} \tag{2}$$

The solution of Eq. 2 for Fig. 3 is

$$\frac{c}{c_o} = \frac{1}{2} \operatorname{erfc}\left(\frac{y}{2\sqrt{\frac{kx}{v}}}\right) \tag{3}$$

where c_o is the concentration of the dye solution. Equation 3 implies that the observed points in Fig. 5 should fall on a straight line. This has been observed to be the case. According to Eq. 3, the distance Δy between the points with $c/c_o = 0.92$ and 0.08 respectively at a cross-section is given by

$$\frac{\Delta y}{\sqrt{\frac{kx}{v}}} = 4 \tag{4}$$

With observed Δy and v in each test, and the value of x at the outlets, the value of k can be computed from Eq. 4.

The values of k thus determined are shown in Fig. 6 where D is the 50% size of the material and ν is the value for the observed temperature of the water. In Fig. 6 are also included the data obtained by Harleman and Rumer[1], with a bed of spheres, and those obtained by Simpson[6] with a mixture of epoxy resin and crushed quartz. The size distributions of their materials are shown in Fig. 1.

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[6] Simpson, E. S., "Transverse Dispersion in Liquid Flow Through Porous Media", U. S. Geol. Survey, Prof. Paper 411-C, 1962.

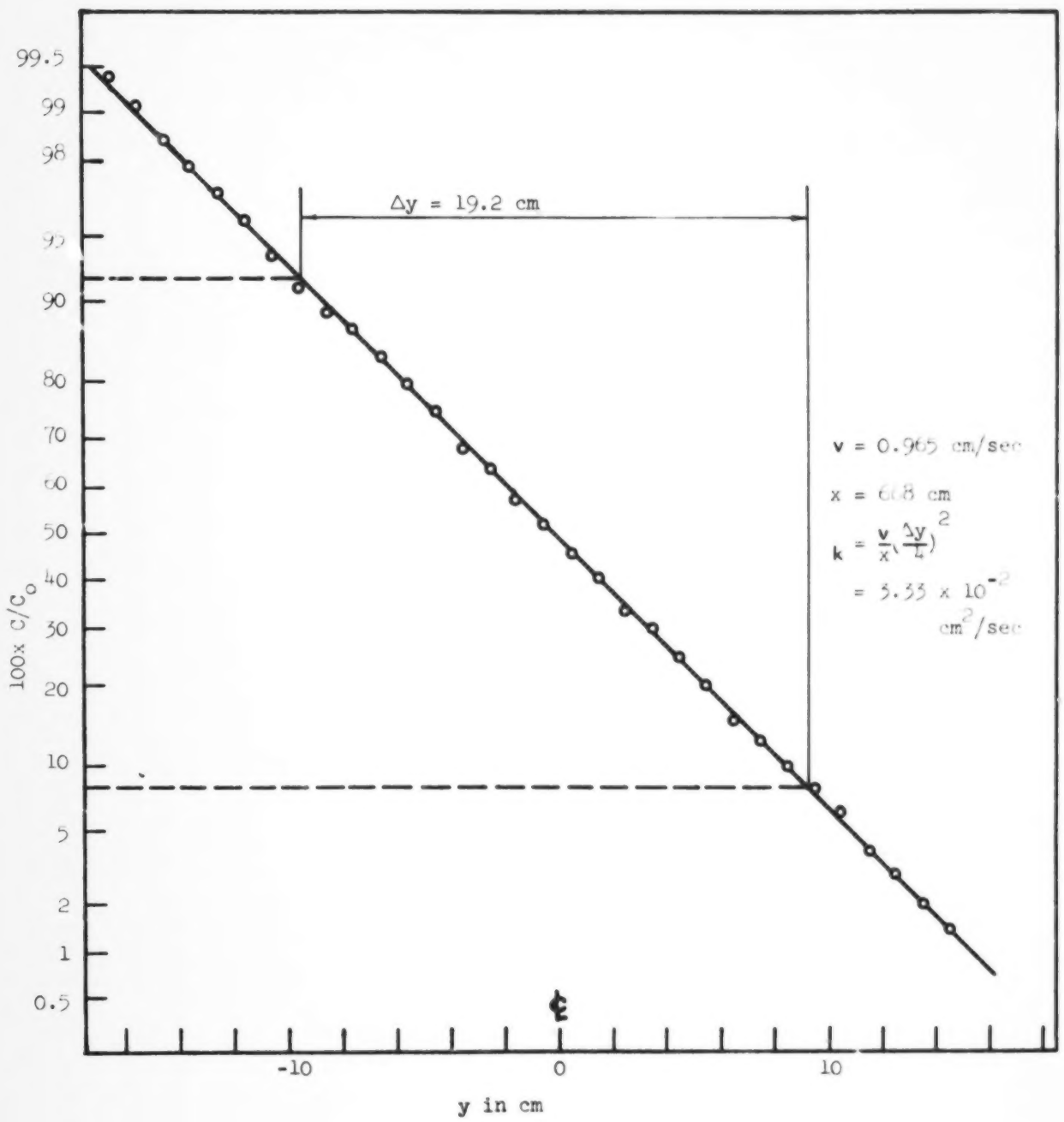

Fig. 5. Typical concentration distribution at the outlets.

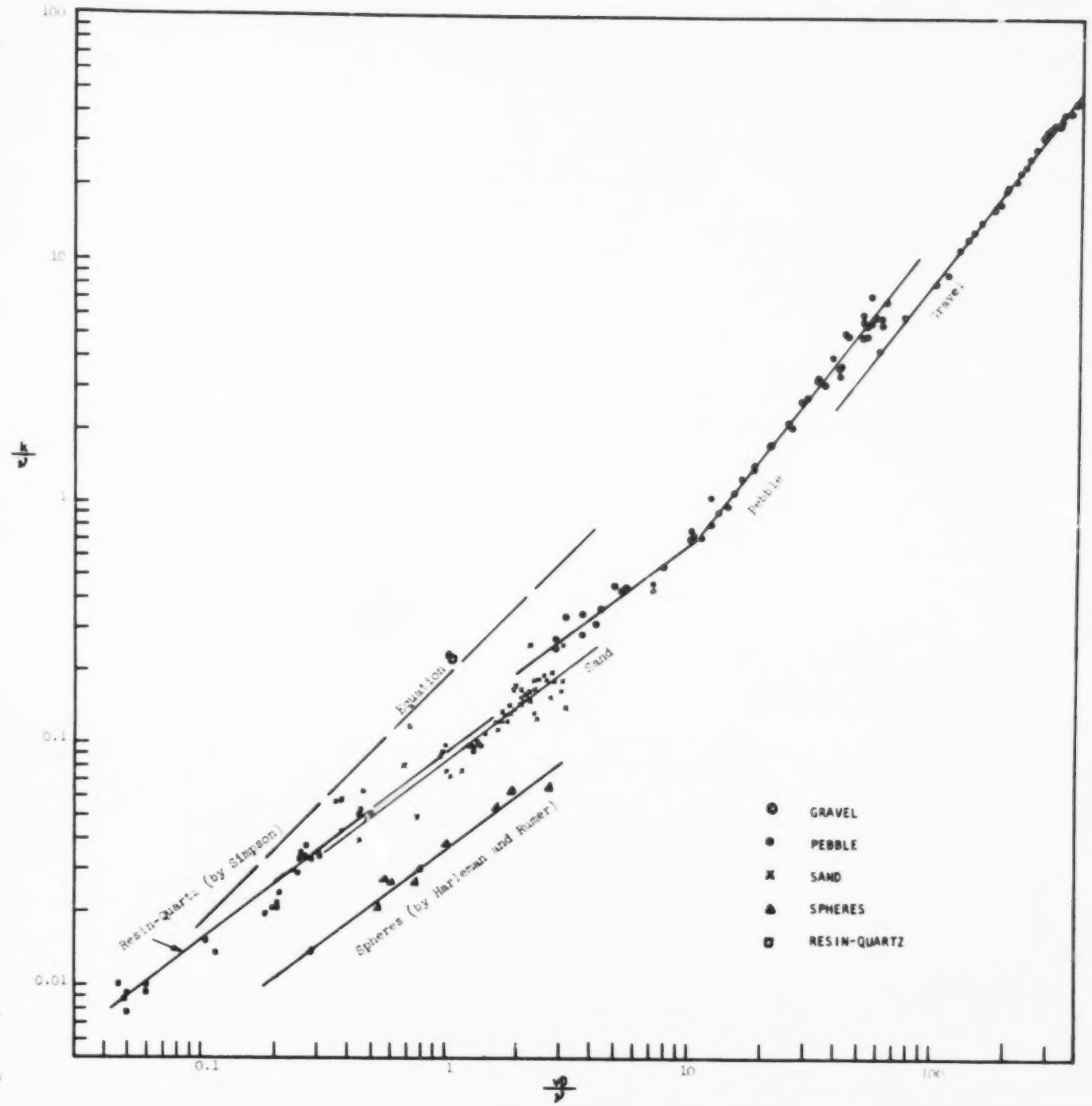

Fig. 6. Observed Coefficient of Transverse Dispersion.

c. Significance of Experimental Results

Since the coefficient k of transverse dispersion must depend on the seepage velocity v, the density ρ and the viscosity μ of the fluid, the average grain size D, and linear measurements d_1, d_2, . . . for the detail description of the grains, we have by dimensional analysis

$$\frac{k}{vD} = f\left(\frac{vD\rho}{\mu}, \frac{D}{d_1}, \frac{D}{d_2}, \ldots\right) \tag{5}$$

where D/d_1, D/d_2, . . . describe such dimensionless geometrical parameters of the porous material as the uniformity of size distribution, the roundness of the grains, porosity, etc. As can be seen from Fig. 6, Eq. 5 may be represented by

$$\frac{k}{\nu} = a_1\left(\frac{vD}{\nu}\right)^{3/4} \quad \text{or} \quad \frac{k}{vD} = a_1\left(\frac{vD}{\nu}\right)^{-1/4} \tag{6}$$

when vD/ν is less than 10 (where $\nu = \mu/\rho$). The value of the coefficient a_1 depends on the dimensionless geometrical parameters of the material. There is a rapid change of slope of the graph near $vD/\nu = 10$. When vD/ν is larger than 10, the data can be represented by

$$\frac{k}{\nu} = a_2\left(\frac{vD}{\nu}\right)^{4/3} \quad \text{or} \quad \frac{k}{vD} = a_2\left(\frac{vD}{\nu}\right)^{1/3} \tag{7}$$

where the value of a_2 varies with the material. The transition from one slope to another in Fig. 6 is amazingly short, and is apparently due to a marked increase of the importance of the inertial forces when vD/ν is larger than 10.

It is interesting to compare the experimental data with an existing statistical theory of dispersion. De Jong[7] and Saffman[8] independently studied

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[7] de Jong, G. de J., "Longitudinal and Transverse Diffusion in Granular Deposits" Trans. Amer. Geophys. Union, Vol. 9, No. 1, 1958.

[8] Saffman, P. G., "A Theory of Dispersion in a Porous Medium", Journ. Fluid Mech., Part 3, Vol. 6, Oct. 1959.

dispersion with an assemblage of randomly orientated tubes as a model of the porous medium. Assuming that the probability distribution function of the displacement of the fluid particles is independent of the Reynolds number and the geometrical parameters of the medium, they obtained for a flow obeying Darcy's law (vD/ν less than about 10)

$$\frac{k}{\nu} = \frac{3}{16}\left(\frac{v\ell}{\nu}\right) \quad \text{or} \quad \frac{k}{v\ell} = \frac{3}{16} \tag{8}$$

where ℓ is a length approximately equal to the grain size. Since the flow paths of the fluid particles are assumed to be independent of the velocity, it is no surprise that the coefficient k is found to be proportional to the velocity v. (For example, with k proportional to v in Eq. 3, the concentration distribution and therefore the flow paths of the contaminant become independent of the velocity). As can be seen from Eq. 5 obtained by dimensional analysis, k being proportional to v implies that density, ρ, and therefore inertial forces, have no influence on the dispersion (and therefore the flow paths) of the contaminant. However, the experimental data indicate that k is not proportional to v. This implies that the flow paths of the contaminant among the pores vary with the Reynolds number due to the influence of inertial forces, even in a flow obeying Darcy's law. This fact should not be considered surprising since Darcy's law is only an approximation when the inertial effect on the pressure gradient is not observable. Actually, inertial forces are there which exert an observable influence on the flow pattern among the pores. (For example, for laminar flow of speed U around a sphere of radius a, the stream function $\Psi(\theta,r)$ near the sphere is[9]

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[9] van Dike, M., Perturbation Methods in Fluid Mechanics, Academic Press, New York, 1964, p. 159.

$$\frac{\Psi}{Ua^2} = \frac{1}{4}\left(\frac{r}{a} - 1\right)^2 \sin^2\theta \left[\left(1 + \frac{3}{8}\frac{Ua}{\nu}\right)\left(2 + \frac{a}{r}\right) - \frac{3}{8}\frac{Ua}{\nu}\left(2 + \frac{a}{r} + \frac{a^2}{r^2}\right)\cos\theta\right] \quad (9)$$

which varies with the Reynolds number Ua/ν.) This small change in the flow pattern with the Reynoids number can be significant in dispersion although the accompanying inertial effect on the pressure gradient is negligible. It seems that an improvement on any statistical theory is to include the influence of the Reynolds number on the probability distribution function of the displacement of the fluid particles.

The coefficient of dispersion k is also influenced by the dimensionless geometrical parameters of the material. With the limited number of materials used in obtaining the data in Fig. 6, no precise conclusion can be drawn as to the relative importance of these parameters. However, it seems that the more angular are the grains, the higher are the values of a_1 and a_2 in Eqs. 6 and 7 respectively. The values a_1 for the pebble, the quartz mixture, the sand and the spheres (listed here in order of decreasing angularity) are 0.12, 0.09, 0.08, and 0.04 respectively. The value of a_2 is 0.03 for the pebble, and is 0.02 for the less angular gravel.

In conclusion, it may be stated that this experimental study has furnished for engineering use the data of the coefficient k of transverse dispersion over a wide range of the Reynolds number, and reveals that inertial forces and dimensionless geometrical parameters are important in the phenomenon of dispersion.

d. Solution for Dispersion From a Line Source

(The materials in Sections d to g are included in a paper entitled "Dispersion from Sources in Non-uniform Seepage Flow" In the May 1967 issue of Journal of Hydraulics Division, Proc. Amer. Soc. of Civil Engrs.)

The solution of the dispersion equation in Eq. 1 for an arbitrary

velocity field is usually very difficult. The objective of this portion of the investigation is to seek solutions of Eq. 1 which are simple and yet adequate for engineering purposes. In this section, we consider the steady two-dimensional dispersion from a line source in a non-uniform flow.

In the dispersion from a source, the contaminant is carried along the streamlines while it disperses across the streamlines. (See First Progress Report of this investigation, or Reference 10). It can be shown that, except within a few grains from the source, longitudinal dispersion is negligible.[11] The width of the trail of contaminant is in general very small compared with its length[10] except where the streamlines diverge rapidly (e.g., near a stagnation point). As long as the flow is such that the width of the trail is small compared with its length and its radius of curvature, the coordinates s along the streamline of contamination, and n normal to this streamline can be used to advantage (See Fig. 7). These coordinates are similar to the boundary-layer coordinates, and can be treated locally as rectangular coordinates.[12]

When coordinates of the boundary-layer type can be used, Eq. 1 for steady two-dimensional lateral dispersion from a line source becomes

$$\frac{\partial(cv_s)}{\partial s} + \frac{\partial(cv_n)}{\partial n} = \frac{\partial}{\partial n}\left(k \frac{\partial c}{\partial n}\right) \tag{10}$$

where v_s and v_n are the s- and n-components respectively of the seepage velocity. The change of density due to contamination is assumed negligible. By definition, v_n is zero along the streamline of contamination. Since the trail of contaminant

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[10] Li, W. H., "Band Diffusion with Variable Parameters along Flow", Proc. ASCE, Vol. 90, No. EM5, 1964.

[11] Hoopes, J. A. and Harleman, D. R. F., "Discussion on Band Diffusion", Proc. ASCE, Vol. 91, No. EM3, 1965.

[12] Li, W. H., and Lam, S. H., Principles of Fluid Mechanics, Addison-Wesley Co., Reading, Mass., 1964, p. 235.

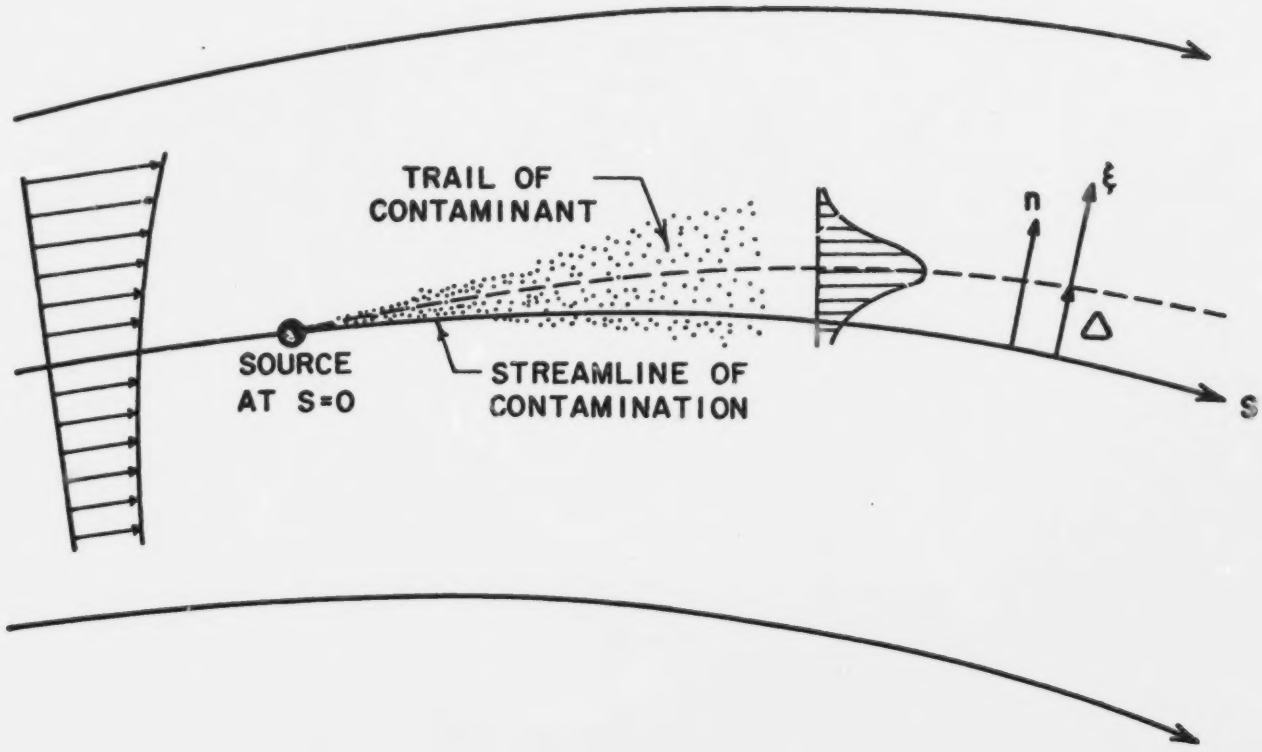

Fig. 7. Two-Dimensional Dispersion from a Line Source.

is relatively narrow, the value of v_n in the trail can be expected to be usually small. We limit our discussion to cases where v_n in the trail is so small that

$$\frac{\partial(cv_n)}{\partial n} \ll \frac{\partial(cv_s)}{\partial s} \tag{11}$$

and

$$v_s \doteq v \tag{12}$$

It can be estimated that these conditions are satisfied if, within the trail at any section s,

$$\frac{|v_n|}{v} \ll \frac{w}{s} \tag{13}$$

where w is any conveniently defined half-width of the trail. Since the maximum $|v_n|$ in the trail is approximately $w\left|\partial v_n/\partial n\right|$, Eq. 13 may be written as

$$\frac{s}{v}\left|\frac{\partial v_n}{\partial n}\right| \ll 1 \tag{14}$$

Under this condition, Eq. 10 becomes

$$\frac{\partial(cv)}{\partial s} = \frac{\partial}{\partial n}\left(k \frac{\partial c}{\partial n}\right) \tag{15}$$

According to Eqs. 6 and 7, k is related to v by

$$\frac{k}{\nu} = a \left(\frac{vD}{\nu}\right)^b \tag{16}$$

where the value of b has been found to be 3/4 when vD/ν is less than 10, and is 4/3 when vD/ν is larger than 10. The coefficient a is of the order of 1/10 for $vD/\nu < 10$, and of the order of 1/40 for $vD/\nu > 10$. Thus for a given flow, both v and k in Eq. 15 are known functions of s and n.

In order to obtain a solution of Eq. 15, simple and yet adequate for practical purposes, three approximations are used: (i) k is considered to be nearly proportional to v; i.e., b is approximately equal to unity in Eq. 16;

(ii) the soil properties and the character of flow (vD/ν smaller or larger than 10) within the narrow trail of contaminant vary only in the s-direction; and (iii) the variation of velocity across the trail can be approximated by

$$v(s,n) \doteq V(s)\, e^{R(s)n} \tag{17}$$

For a given flow v(s,n), use

$$V(s) = v(s,0) \tag{18}$$

i.e., V(s) is the velocity distribution along the streamline of contamination. The quantity R(s) is related to the lateral rate of change of v at this streamline: Expand Eq. 17 with respect to n.

$$v(s,0) + \left.\frac{\partial v}{\partial n}\right|_{n=0} n + \ldots \doteq V(s)\left[1 + R(s)n + \ldots\right] \tag{19}$$

Thus, at any s,

$$R(s) = \left.\frac{1}{V}\frac{\partial v}{\partial n}\right|_{n=0} \tag{20}$$

It will be demonstrated subsequently in Section e that the errors due to these approximations can usually be tolerated in practical cases.

With these approximations, the solution of Eq. 15 for a line source at s = 0 will be shown presently to be

$$cv = \frac{N}{\sqrt{4\pi\,\overline{\left(\frac{K}{V}\right)}\,s}} \exp\left[-\frac{\xi^2}{4\,\overline{\left(\frac{K}{V}\right)}\,s}\right] \tag{21}$$

where K(s) is the value of k along the streamline of contamination, and is, according to Eqs. 16 and 18,

$$K(s) = a\, D^b V^b \nu^{1-b} \tag{22}$$

and

$$\overline{\left(\frac{K}{V}\right)} = \frac{1}{s}\int_0^s \frac{K}{V}\, ds \tag{23}$$

is the average value of K/V from the source up to the section of interest at s. The quantity N in Eq. 21 is the strength of the line source in mass per unit length per unit time. The distance ξ is (See Fig. 7)

$$\xi = n - \Delta \tag{24}$$

where

$$\Delta(s) = \int_0^s \frac{KR}{V}(2-b)\, ds \tag{25}$$

The solution in Eq. 21 can be obtained as follows: Eq. 15 can be written as

$$\frac{\partial(cv)}{\partial s} = \frac{\partial k}{\partial n}\frac{\partial c}{\partial n} + k\frac{\partial^2 c}{\partial n^2} \tag{26}$$

A solution is sought for the following conditions at s = 0:

$$c = 0 \text{ at } n \neq 0 \tag{27}$$

$$\int_\infty^\infty cv\, dn = N \tag{28}$$

Since

$$\frac{\partial(cv)}{\partial n} = v\frac{\partial c}{\partial n} + c\frac{\partial v}{\partial n} \tag{29}$$

$$\frac{\partial^2(cv)}{\partial n^2} = v\frac{\partial^2 c}{\partial n^2} + \frac{2}{v}\frac{\partial v}{\partial n}\left[\frac{\partial(cv)}{\partial n} - c\frac{\partial v}{\partial n}\right] + c\frac{\partial^2 v}{\partial n^2} \tag{30}$$

Eq. 26 can be written as

$$\frac{v}{k}\frac{\partial(cv)}{\partial s} = \frac{\partial^2(cv)}{\partial n^2} + \left[\frac{1}{k}\frac{\partial k}{\partial n} - \frac{2}{v}\frac{\partial v}{\partial n}\right]\frac{\partial(cv)}{\partial n} + \left[2\left(\frac{1}{v}\frac{\partial v}{\partial n}\right)^2 - \frac{1}{v}\frac{\partial^2 v}{\partial n^2} - \frac{1}{kv}\frac{\partial k}{\partial n}\frac{\partial v}{\partial n}\right](cv) \tag{31}$$

With the approximations (ii) and (iii), Eqs. 16 and 22 give

$$k(s,n) = K(s)\, e^{bRn} \tag{32}$$

Eq. 31 becomes

$$\frac{V}{K}\frac{\partial(cv)}{\partial s} e^{(1-b)Rn} = \frac{\partial^2(cv)}{\partial n^2} - (2-b)R\frac{\partial(cv)}{\partial n} + (1-b)R^2(cv) \tag{33}$$

In view of approximation (i), we neglect terms of the order of (1-b). Then

$$\frac{V}{K}\frac{\partial(cv)}{\partial s} = \frac{\partial^2(cv)}{\partial n^2} - (2 - b)R\frac{\partial(cv)}{\partial n} \tag{34}$$

Change independent variables to

$$\eta = \int_0^s \frac{K}{V}\,ds \tag{35}$$

$$\xi = n - \Delta(s) \tag{36}$$

where $\Delta(s)$ is to be chosen subsequently. Eq. 34 becomes

$$\frac{\partial(cv)}{\partial \eta} = \frac{\partial^2(cv)}{\partial \xi^2} + \left[\frac{V}{K}\frac{d\Delta}{ds} - (2 - b)R\right]\frac{\partial(cv)}{\partial \xi} \tag{37}$$

Choose Δ as described by Eq. 25 so that Eq. 37 becomes

$$\frac{\partial(cv)}{\partial \eta} = \frac{\partial^2(cv)}{\partial \xi^2} \tag{38}$$

The solution of this equation for the conditions in Eqs. 27 and 28 is

$$cv = \frac{N}{\sqrt{4\pi\eta}}\exp\left(-\frac{\xi^2}{4\eta}\right) \tag{39}$$

Since $\eta = \overline{(K/V)}\,s$, this is the solution in Eq. 21.

It is interesting to compare the case of non-uniform flow to the case of uniform flow. With constant v and k, and R = 0, Eq. 21 gives for the uniform flow

$$cv = \frac{N}{\sqrt{4\pi\frac{k}{v}s}}\exp\left(-\frac{n^2}{4\frac{k}{v}s}\right) \tag{40}$$

Comparison of Eqs. 21 and 40 shows that, in the case of non-uniform flow, the longitudinal scale and the maximum cv at a section are modified due to the change from k/v to $\overline{(K/V)}$, and the trail of contaminant is shifted through a distance Δ given by Eq. 25 (See Fig. 7). The distribution of c is not symmetrical when v varies with n or ξ.

e. Adequacy of Eq. 21 for practical use

The solution in Eq. 21 for a line sources in a non-uniform flow has been obtained with three approximations. It is necessary to examine the errors caused by these approximations.

(i) Effect of $b \neq 1$:

When the velocity can be approximated by Eq. 17, but $b \neq 1$, the dispersion equation, Eq. 15, becomes Eq. 33 which may be written as

$$\frac{v}{k}\frac{\partial(cv)}{\partial s} = \frac{\partial^2(cv)}{\partial n^2} - (2 - b)R\frac{\partial(cv)}{\partial n} + (1 - b)R^2(cv) \qquad (41)$$

Eq. 21 is an exact solution of this equation when $b = 1$. To reveal the error of Eq. 21 for $b \neq 1$, Eq. 21 is to be compared with numerical solutions of Eq. 41 obtained with a digital computer for several values of b between 3/4 and 4/3. (For details of the numerical solution, see Reference 13.)

A six-point implicit numerical method is chosen for the numerical solution because this method can yield, after twenty steps along the flow, a solution of Eq. 41 with $b = 1$ with the error in c at any point no larger than 0.5% of the maximum c at the cross section as compared with the exact solution in Eq. 21. The numerical solutions for $b = 3/4$ and $4/3$ for a given flow are shown in Fig. 8, and compared with the corresponding solutions obtained from Eq. 21. Note that, in this flow, there is about 500% lateral variation of v across the half-width of the trail of contaminant. Although this flow may not be realistic, it serves as a severe test of Eq. 21 as an approximate solution of Eq. 41 when $b \neq 1$. For less extreme cases, the solutions obtained by numerical method and from Eq. 21 practically coincide. Thus it is evident that Eq. 21 is adequate for practical use with b between 3/4 and 4/3.

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[13] Canale, R.P., "Dispersion from a Line Source in a Non-uniform Flow through a Porous Medium", M.S. thesis, Dept. of Civ. Engrg., Syracuse Univ., 1966.

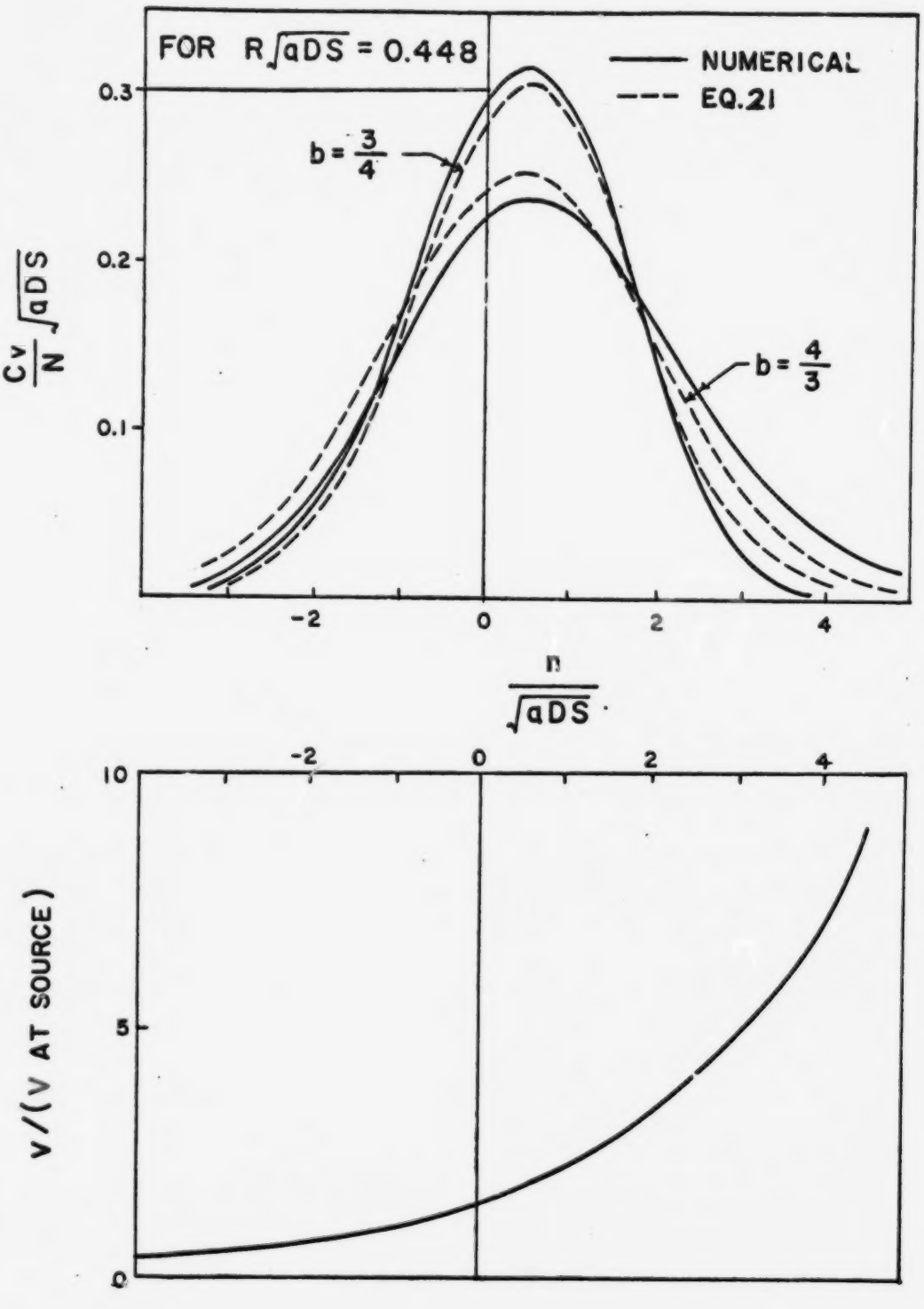

Fig. 8. Accuracy of Eq. 21 for $b = \frac{3}{4}$ and $\frac{4}{3}$.

(ii) Effect of lateral variation of v and k:

To test the adequacy of Eq. 21 as the solution of Eq. 15 when the velocity distribution is not exponential in n, we consider first the circular flow shown in Fig. 9 with the velocity distribution

$$v = \frac{VL}{r} \qquad (42)$$

where V is the constant velocity along the streamline of contamination at r = L. In cylindrical coordinates r and θ, Eq. 1 for steady two-dimensional lateral dispersion in this flow becomes

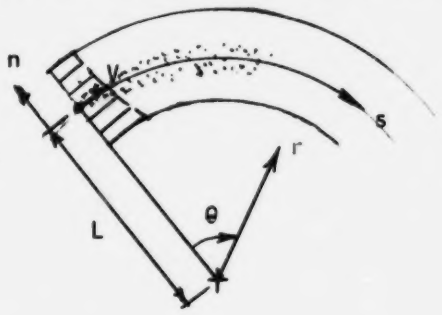

Fig. 9. Dispersion in Flow with Circular Streamlines.

$$\frac{1}{r}\frac{\partial(cv)}{\partial\theta} = \frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial c}{\partial r}\right) \qquad (43)$$

The numerical solution of this equation is to be compared with Eq. 21 for b = 1. With b = 1, we have k = aDv = aDVL/r. Since s= Lθ and n = r - L, Eq. 43 becomes

$$\frac{L}{aD(L+n)}\frac{\partial c}{\partial s} = \frac{\partial^2 c}{\partial n^2} \qquad (44)$$

For Eq. 21, we have from Eq. 20

$$R = \frac{1}{V}\frac{\partial v}{\partial r}\bigg|_{r=L} = -\frac{1}{L} \qquad (45)$$

K = aDV, and Δ = -aDs/L; and Eq. 21 becomes

$$cv = \frac{N}{\sqrt{4\pi aDs}}\exp\left[-\frac{1}{4}\left(\frac{n}{\sqrt{aDs}} + \frac{\sqrt{aDs}}{L}\right)^2\right] \qquad (46)$$

At $\sqrt{aDs}/L = 1/30$, the maximum error in c of Eq. 46 is $1\frac{1}{2}\%$ of the maximum c at the section. At this section, the width of the trail is about 20% of the radius L of the streamline of contamination.

The flow shown in Fig. 10a has also been used for the comparison of Eq. 21 and numerical solutions. (For the use of Eq. 21 in this case, see Section f below.) It is found that the two solutions of c at any point differ by less than 0.2% of the maximum c at the section.

It is thus established that Eq. 21 is adequate for practical use as long as the radius of curvature of the streamline of contamination is large compared with the width of the trail and Eq. 14 is satisfied on this streamline.

f. Contamination of well by dispersion

An example of application of Eq. 21 to the flow described by Eq. 42 has been given above. Here the more complicated but more realistic problem of the contamination of a well by dispersion is considered. In Fig. 10a is shown a well drawing from a uniform ground-water flow. This flow is described by the following stream function and velocity components:[14]

$$\Psi = \frac{Q}{2\pi P} \cot^{-1}\left(\frac{x}{y}\right) - Uy \tag{47}$$

$$v_x = U - \frac{Q}{2\pi P} \frac{x}{(x^2 + y^2)} \tag{48}$$

$$v_y = -\frac{Q}{2\pi P} \frac{y}{(x^2 + y^2)} \tag{49}$$

where Ψ is the stream function, Q represents the discharge per unit length of the well, P denotes the porosity of the soil, U is the seepage velocity of the approaching flow, and v_x and v_y are the x- and y-components respectively of the seepage velocity v. In the following computation, all quantities should be in consistent units; e.g., ft.-lb.-sec. units. First, accurate results are obtained by using the stream function. A simplified procedure for engineering estimate will then be presented.

[14] Li, W.H. and Lam, S.H., Principles of Fluid Mechanics, Addison-Wesley Co., Reading, Mass., 1964, Chap. 7.

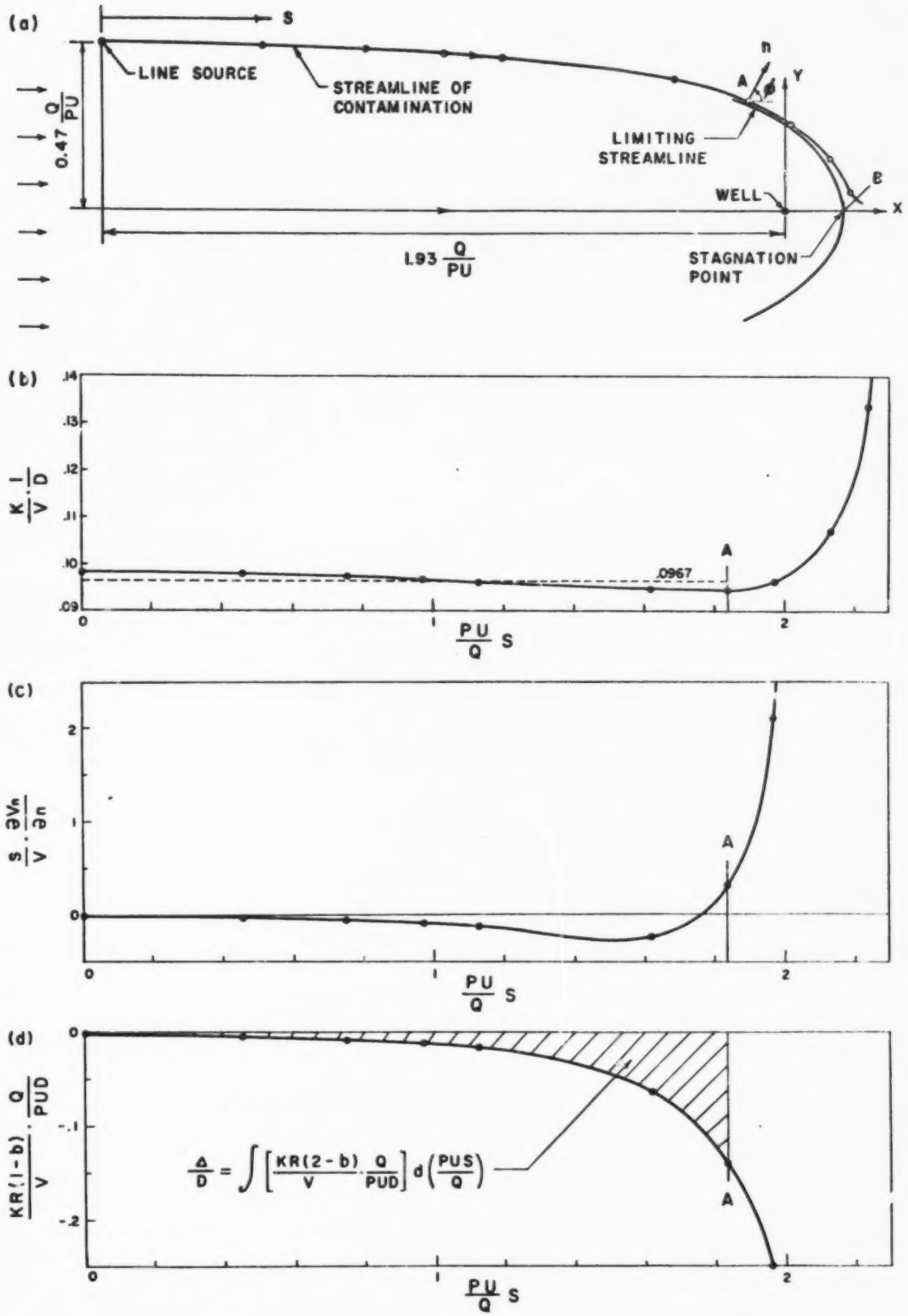

Fig. 10. Contamination of Well by Dispersion.

(i) Streamline of contamination:

For a line source located as shown in Fig. 10a, the value of ψ for the streamline of contamination is

$$\psi = \frac{Q}{2\pi P} \cot^{-1}\left(\frac{-1.93}{0.47}\right) - 0.47 \frac{Q}{P} = -0.008 \frac{Q}{P} \tag{50}$$

With this value of ψ in Eq. 47, we have the equation of this streamline:

$$x = y \cot\left[2\pi \left(\frac{PU}{Q} y - 0.008\right)\right] \tag{51}$$

This streamline can now be plotted point by point, and s of each point can be measured from the plot, as shown in Fig. 10a. The equation of the limiting streamline is obtained with $\psi = 0$ in Eq. 47, and the streamline can be plotted in a similar manner.

The slope of the streamline at any point is $dy/dx = v_y/v_x$. The angle ϕ indicating the n-direction at a point on the streamline of contamination is given by

$$\tan \phi = -\frac{1}{\left(\frac{dy}{dx}\right)} = -\frac{v_x}{v_y} \tag{52}$$

On this streamline, we have

$$V = \sqrt{v_x^2 + v_y^2} \tag{53}$$

The value of K at each point is computed from Eq. 22. In this example, we use $a = 1/10$, $b = 3/4$, and $UD/\nu = 1$. Then

$$\frac{K}{VD} = \frac{1}{10}\left(\frac{UD}{\nu}\frac{V}{U}\right)^{-1/4} = \frac{1}{10}\left(\frac{V}{U}\right)^{-1/4} \tag{54}$$

For example, take point A at $y = 0.31$ Q/PU. We have $x = -0.105$ Q/PU from Eq. 51, $s = 1.837$ Q/PU from Fig. 10a, $v_x = 1.156$ U from Eq. 48, $v_y = -0.462$ U from Eq. 49, $\phi = 68\frac{1}{4}^{\circ}$ from Eq. 52, $V = 1.245$ U from Eq. 53, and $K/V = 0.0947$ D from Eq. 54. The computed values of K/V at several points on the streamline of

contamination are plotted in Fig. 10b.

(ii) Velocity variation:

In order to compute R and to examine the condition in Eq. 14, we first compute the spatial derivatives of v_x and v_y. By differentiating Eqs. 48 and 49, we obtain for point A $\partial v_y/\partial y = -\partial v_x/\partial x = 1.18\ PU^2/Q$, and $\partial v_x/\partial y = \partial v_y/\partial x = -0.903\ PU^2/Q$.

Now, with respect to any point such as A, we have

$$v_n = v_x \cos\phi + v_y \sin\phi \tag{55}$$

$$v_s = v_x \sin\phi - v_y \cos\phi \tag{56}$$

and the following relationship among the operators:

$$\frac{\partial}{\partial n} = \frac{\partial x}{\partial n}\frac{\partial}{\partial x} + \frac{\partial y}{\partial n}\frac{\partial}{\partial y} = \cos\phi \frac{\partial}{\partial x} + \sin\phi \frac{\partial}{\partial y} \tag{57}$$

Thus

$$\frac{\partial v_n}{\partial n} = \cos^2\phi \frac{\partial v_x}{\partial x} + \sin\phi\cos\phi \left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right) + \sin^2\phi \frac{\partial v_y}{\partial y} \tag{58}$$

$$\frac{\partial v_s}{\partial n} = \sin\phi\cos\phi \left(\frac{\partial v_x}{\partial x} - \frac{\partial v_y}{\partial y}\right) - \cos^2\phi \frac{\partial v_y}{\partial x} + \sin^2\phi \frac{\partial v_x}{\partial y} \tag{59}$$

For example, at point A, we have from Eqs. 58 and 59, $\partial v_n/\partial n = 0.234\ PU^2/Q$ and $\partial v_s/\partial n = -1.47\ PU^2/Q$.

To see whether Eq. 14 is satisfied as assumed, we compute $\left|\partial v_n/\partial n\right|_s/V$. For point A, this value is $0.234 \times 1.837/1.245 = 0.345$. The values for other points along the streamline of contamination are shown in Fig. 10c. The condition of Eq. 14 is met satisfactorily at the points between the source and point A. Downstream of point A, Eq. 14 is not satisfied as the streamlines begin to diverge rapidly, and Eq. 21 can only yield a rough estimate of the concentration distribution.

(iii) Concentration distribution:

To compute the shift Δ in Eq. 25, first compute R from Eq. 20. For point A, $R = (-1.47/1.245)PU/Q = -1.18\ PU/Q$. Then $(2-b)RK/V = (2-\frac{3}{4}) \times (-1.18) \times 0.0947\ PUD/Q = -0.139\ PUD/Q$. The value of $(2-b)RK/V$ along the streamline of contamination is plotted in Fig. 10d. The integral of this quantity between the source and point A (the shaded area in Fig. 10d) gives $\Delta = -0.05D$ at point A (see Eq. 25). It can be seen that this shift Δ is negligibly small.

To compute the concentration distribution over the cross section at point A, first compute the mean value of K/V in the stretch between the source and point A. This mean value is $\overline{(K/V)} = 0.0967\ D$ as shown in Fig. 10b. With this value of $\overline{(K/V)}$, $s = 1.837\ Q/PU$, and $\Delta = -0.05D$, we have from Eq. 21 for section A,

$$cv = 0.67 \frac{PUN}{Q} \sqrt{\frac{Q}{PUD}} \exp\left[-1.41 \frac{Q}{PUD} \left(\frac{PUn}{Q} + 0.05 \frac{PUD}{Q}\right)^2\right] \tag{60}$$

This equation is plotted in Fig. 11 for the case with $\frac{Q}{PUD} = 10^4$.

The limiting streamline at section A can be shown to be located at $n = -0.0066\ Q/PU$ (see Fig. 10a). The amount of contaminant which flows below the limiting streamline (and will eventually flow into the well) at this section is indicated by the shaded area in Fig. 11, and is given by the following integral:

$$\int_{-\infty}^{-0.0066Q/PU} cv\ dn = \frac{0.67\ N}{\sqrt{1.41}} \int_{-\infty}^{-0.66\sqrt{1.41}} \exp\left[-\left(10^2\sqrt{1.41}\ \frac{PUn}{Q}\right)^2\right] d\left(10^2\sqrt{1.41}\ \frac{PUn}{Q}\right)$$

$$= \frac{0.67\sqrt{\pi}\ N}{2\sqrt{1.41}} \left[1 - \mathrm{erf}(0.66\sqrt{1.41})\right] = 0.134\ N \tag{61}$$

(Here Eq. 60 has been used with $\frac{Q}{PUD} = 10^4$, and the small quantity $0.05 \frac{PUD}{Q}$ has been neglected.) Therefore 13.4 percent of the contaminant is passing below the limiting streamline at section A.

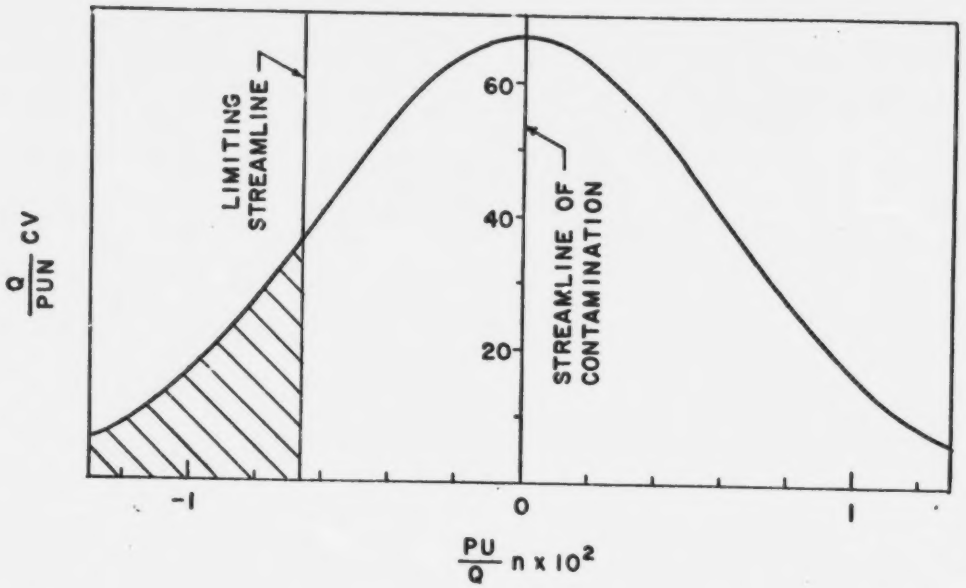

Fig. 11. Concentration Distribution at Section A in Fig. 10(a).

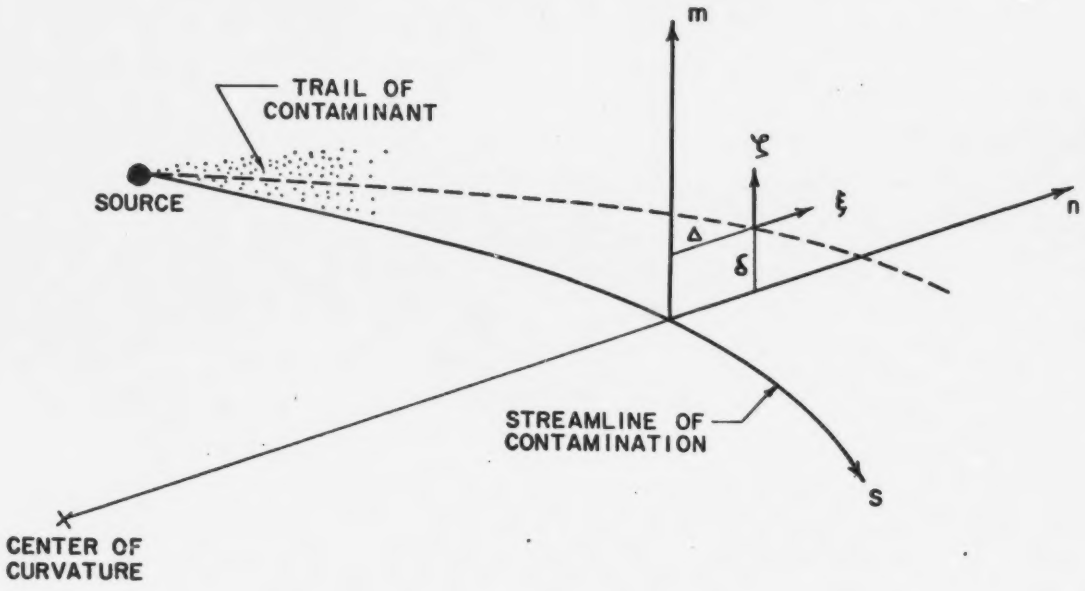

Fig. 12. Dispersion from a Point Source.

The percentage of contaminant flowing into the well should be computed at section B passing the stagnation point (see Fig. 10a). However, Eq. 21 is not intended for use near section B where the streamlines diverge rapidly. Use of Eq. 21 at sections downstream of point A can yield only approximate results.

(iv) Engineering estimate:

In engineering estimate, the detailed computation shown above is not necessary. The values of $\partial v_n/\partial n$ on the streamline of contamination should first be estimated at points where the streamlines begin to diverge or converge rapidly. These values will show where Eq. 14 is satisfied and Eq. 21 is applicable. It is sufficient to assume b = 1 in engineering estimate. Then K = aDV and $\overline{(K/V)}$ = aD. From the estimated values of $(\partial v_s/\partial n)/V$ at a few points along the streamline of contamination, compute the mean value and use this value for R to compute the small shift Δ at a section at s: $\Delta \doteq$ RaDs. The from Eq. 21,

$$cv^* \doteq \frac{N}{\sqrt{4\pi aDs}} \exp\left[-\frac{(n - RaDs)^2}{4aDs}\right] \tag{62}$$

g. Solution for dispersion from a point source

A solution for the steady dispersion from a point source has also been obtained by the same method. In this case, the equation corresponding to Eq. 15 is

$$\frac{\partial(cv)}{\partial s} = \frac{\partial}{\partial n}\left(k \frac{\partial c}{\partial n}\right) + \frac{\partial}{\partial m}\left(k \frac{\partial c}{\partial m}\right) \tag{63}$$

where n is directed from the center of curvature of the streamline of contamination, and m-direction is normal to n- and s-directions as shown in Fig. 12. A solution is sought with the same approximations (i) and (ii) as with Eq. 15 in Section d, and condition (iii) that the velocity across the trail can be approximated as

$$v(s,n,m) \triangleq V(s)\, e^{R(s)n}\, e^{R'(s)m} \tag{64}$$

For a given $v(s,n,m)$, use

$$V(s) = v(s,0,0) \tag{65}$$

$R(s)$ from Eq. 20, and

$$R'(s) = \frac{1}{V}\frac{\partial v}{\partial m}\bigg|_{m=0} \tag{66}$$

As will be shown presently, the solution of Eq. 63 for a point source is

$$cv = \frac{M}{4\pi\left(\overline{\frac{K}{V}}\right) s} \exp\left[-\frac{\xi^2 + \zeta^2}{4\left(\overline{\frac{K}{V}}\right) s}\right] \tag{67}$$

where M is the strength of the point source in mass per unit time, $\overline{(\frac{K}{V})}$ and ξ have been defined in Eqs. 23 and 24 respectively, and

$$\zeta = m - \delta \tag{68}$$

and

$$\delta(s) = \int_0^s \frac{KR'}{V}(2-b)\, ds \tag{69}$$

Thus the trail of contaminant is displaced through δ in the m-direction as well as through Δ in the n-direction as shown in Fig. 12.

To obtain the solution in Eq. 67, substitute into Eq. 63 $\partial c/\partial n$ and $\partial^2 c/\partial n^2$ from Eqs. 29 and 30, and similar expressions for $\partial c/\partial m$ and $\partial^2 c/\partial m^2$ to get

$$\frac{v}{k}\frac{\partial(cv)}{\partial s} = \frac{\partial^2(cv)}{\partial n^2} + \frac{\partial^2(cv)}{\partial m^2} + \left(\frac{1}{k}\frac{\partial k}{\partial n} - \frac{2}{v}\frac{\partial v}{\partial n}\right)\frac{\partial(cv)}{\partial n} + \left(\frac{1}{k}\frac{\partial k}{\partial m} - \frac{2}{v}\frac{\partial v}{\partial m}\right)\frac{\partial(cv)}{\partial m}$$

$$+ \left[2\left(\frac{1}{v}\frac{\partial v}{\partial n}\right)^2 - \frac{1}{v}\frac{\partial^2 v}{\partial n^2} - \frac{1}{kv}\frac{\partial k}{\partial n}\frac{\partial v}{\partial n} + 2\left(\frac{1}{v}\frac{\partial v}{\partial m}\right)^2 - \frac{1}{v}\frac{\partial^2 v}{\partial m^2} - \frac{1}{kv}\frac{\partial k}{\partial m}\frac{\partial v}{\partial m}\right](cv) \tag{70}$$

With the approximations (ii) and (iii), Eq. 16 and 22 give

$$k(s,n,m) = K(s)\, e^{bRn}\, e^{bR'm} \tag{71}$$

Eq. 70 becomes

$$\frac{V}{K}\frac{\partial(cv)}{\partial s}\, e^{(1-b)(Rn+R'm)} = \frac{\partial^2(cv)}{\partial n^2} + \frac{\partial^2(cv)}{\partial m^2} - (2-b)\left[R\frac{\partial(cv)}{\partial n} + R'\frac{\partial(cv)}{\partial m}\right]$$

$$+ (1-b)(R^2 + R'^2)(cv) \qquad (72)$$

In view of approximation (i), we neglect terms of the order of (1-b). Then

$$\frac{V}{K}\frac{\partial(cv)}{\partial s} = \frac{\partial^2(cv)}{\partial n^2} + \frac{\partial^2(cv)}{\partial m^2} - (2-b)\left[R\frac{\partial(cv)}{\partial n} + R'\frac{\partial(cv)}{\partial m}\right] \qquad (73)$$

Change independent variables to η and ξ defined in Eqs. 35 and 36, and

$$\zeta = m - \delta(s) \qquad (74)$$

Then

$$\frac{\partial(cv)}{\partial \eta} = \frac{\partial^2(cv)}{\partial \xi^2} + \frac{\partial^2(cv)}{\partial \zeta^2} + \left[\frac{V}{K}\frac{d\Delta}{ds} - (2-b)R\right]\frac{\partial(cv)}{\partial \xi} + \left[\frac{V}{K}\frac{d\delta}{ds} - (2-b)R'\right]\frac{\partial(cv)}{\partial \zeta} \qquad (75)$$

Choose Δ and δ as described by Eqs. 25 and 69 respectively, so that Eq. 75 becomes

$$\frac{\partial(cv)}{\partial \eta} = \frac{\partial^2(cv)}{\partial \xi^2} + \frac{\partial^2(cv)}{\partial \zeta^2} \qquad (76)$$

The solution of this equation for a point source of strength M is

$$cv = \frac{M}{4\pi\eta}\exp\left(-\frac{\xi^2 + \zeta^2}{4\eta}\right) \qquad (77)$$

Since $\eta = \overline{\left(\frac{K}{V}\right)}\, s$, we have the solution in Eq. 67.

Eq. 67 is easy to use in the case of a point source in a two-dimensional velocity field. With the m-direction normal to the plane of flow, we have $R' = 0$, $\delta = 0$, and $\zeta = m$. The trail in this case is shifted in the n-direction only.

h. Diffusion from sources with variable parameters along flow

(The material in this section is the summary of a paper and discussion on "Band Diffusion with Variable Parameters along Flow" in the Oct. 1964 and Dec. 1965 issues of Journ. of Eng. Mech. Div., Proc. Amer. Soc. Civ. Engrs.)

Solutions for the diffusion from sources in non-uniform flows have also been obtained for cases where the coefficient k is not necessarily related

to the velocity v as described by Eqs. 6 and 7; e.g., in turbulent diffusion and heat conduction. The following solutions are therefore applicable not only to dispersion in ground water. However, the effects of lateral variation of k and v across the narrow width of the trail of diffusion are neglected in these solutions.

For steady, constant density, two-dimensional flows as shown in Fig.7, Eq. 1 becomes

$$v_s \frac{\partial c}{\partial s} + v_n \frac{\partial c}{\partial n} = k \frac{\partial^2 c}{\partial n^2} \tag{78}$$

By introducing the Lagrange stream function

$$\psi = n v_s \tag{79}$$

with

$$v_n = -\frac{\partial \psi}{\partial s} \tag{80}$$

we have in terms of independent variables s and ψ, instead of s and n,

$$\left.\frac{\partial c}{\partial s}\right|_{\text{const.}n} = -v_n \frac{\partial c}{\partial \psi} + \left.\frac{\partial c}{\partial s}\right|_{\text{const.}\psi} \tag{81}$$

$$\frac{\partial c}{\partial n} = v_s \frac{\partial c}{\partial \psi} \tag{82}$$

Eq. 78 becomes, with $v \doteq v_s$,

$$\frac{1}{kv} \frac{\partial c}{\partial s} = \frac{\partial^2 c}{\partial \psi^2} \tag{83}$$

In the case of diffusion which is axially symmetrical about the s-axis, the equation corresponding to Eq. 83 is

$$\frac{1}{k} \frac{\partial c}{\partial s} = 2\left(\Psi \frac{\partial^2 c}{\partial \Psi^2} + \frac{\partial c}{\partial \Psi}\right) \tag{84}$$

where Ψ is the Stokes stream function

$$\Psi = \frac{1}{2} v r^2 \tag{85}$$

and r is measured radially from the s-axis.

In Eqs. 83 and 84, k and v are assumed to be functions of s only. It can then be shown that the solution of Eq. 83 for a line source of strength N is

$$c = \frac{N}{2\sqrt{\pi\, \overline{kv}\, s}} \exp\left(-\frac{\rho^2}{4\, \overline{kv}\, s}\right) \tag{86}$$

where $\overline{kv}$ is the mean value of kv from the source to the section of interest:

$$\overline{kv} = \frac{1}{s}\int_0^s kv\, ds \tag{87}$$

The solution of Eq. 84 for a point source of strength M is

$$c = \frac{M}{4\pi\, \bar{k}\, s} \exp\left(-\frac{\Psi}{4\, \bar{k}\, s}\right) \tag{88}$$

where $\bar{k}$ is the mean value of k from the source to the section of interest:

$$\bar{k} = \frac{1}{s}\int_0^s k\, ds \tag{89}$$

These solutions are in the same forms as the solutions for constant k and v. Physically, this means that, in a non-uniform flow with variable k and v, the diffusing quantity is carried along the streamlines while it diffuses with an effective value of kv or k which is equal to the mean value $\overline{kv}$ or $\bar{k}$, respectively, up to the section of interest.

i. Dispersion at interface of two miscible liquids in shear flow

(The materials in Sections i and j are included in a paper entitled "Dispersion at the Interface of Miscible Liquids in a Soil" to be published in Water Resources Research of American Geophysical Union.)

Another dispersion problem of engineering interest is that of two miscible fluids in shear flow; e.g., flow of fresh water over salt water in the ground. The two fluids may differ in density and viscosity, and are under the influence of gravity. The shape of the interface has been discussed by others[5,15].

[15] Lin, K.M., "Unsteady Flow Pattern for Transient Two-Phase Flow through Porous Media", presented at 5th U.S. Congress of Appl. Mech., June 1966.

We will study the dispersion at the interface in the flows shown in Figs. 13, 15, and 17. The two liquids involved may be (1) a layer of liquid of density ρ_o, and another layer of the same liquid with a concentration c_o of a tracer material and density $(\rho_o + \Delta\rho)$, e.g., fresh water and sea water; or (2) two miscible liquids of densities ρ_o and $(\rho_o + \Delta\rho)$ respectively, e.g., alcohol and water. (In the latter case, c_o is the concentration of the latter liquid in the soil.)

In the zone of dispersion, the concentration c of the liquid is related to the density ρ by

$$\rho = \rho_o + \frac{\Delta\rho}{c_o} c \tag{90}$$

The flow is assumed to obey Darcy's law:

$$\vec{v} = -\frac{\beta}{\mu}(\nabla p - \rho\vec{g}) \tag{91}$$

where p is pressure, β stands for the intrinsic permeability of the soil, and $\vec{g}$ is the gravitational acceleration. For a macroscopically uniform porous medium, the continuity equation for a steady flow is

$$\nabla.(\rho\vec{v}) = 0 \tag{92}$$

The steady dispersion of the two miscible fluids is described by Eq. 1 as

$$\nabla.(c\vec{v}) = \nabla.\left[\rho T \nabla\left(\frac{c}{\rho}\right)\right] \tag{93}$$

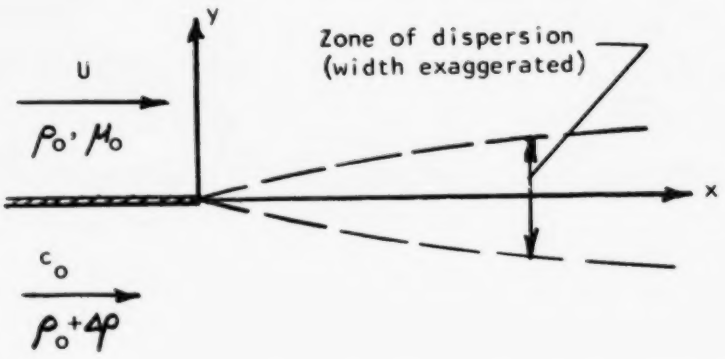

Fig. 13. Dispersion in Two Liquids in Horizontal Parallel Flow.

where T is the tensor coefficient of dispersion. In view of Eqs. 90 and 92, Eq. 93 can be reduced to

$$\vec{v} \cdot \nabla c = \frac{\rho}{\rho_o} \nabla \cdot \left[\rho T \nabla \left(\frac{c}{\rho} \right) \right] \tag{94}$$

It will be seen that lateral dispersion is predominant in the zone of dispersion. The component k of T for lateral dispersion is, according to Eq.6,

$$\frac{\rho k}{\mu} = a_1 \left(\frac{\rho v D}{\mu} \right)^b \tag{95}$$

where the index b is about 3/4 for $\rho vD/\mu <$ [illegible]. The coefficient a_1 depends on the geometrical parameters of the porous medium, and is approximately equal to 0.1 for sands.

Outside the zone of dispersion, Eq. 94 may be written for a two-dimensional flow in the x-y plane as

$$\frac{v_x}{U} \frac{\partial (c/c_o)}{\partial (x/L)} + [\text{illegible}] \frac{\partial (c/c_o)}{\partial (y/L)} = O(e^2) \tag{96}$$

where v_x and v_y are the x- and y-components of the seepage velocity $\vec{v}$, U is a representative value of v, and L is a representative length of the zone of dispersion. On the right side of Eq. 96 are terms of the order of $e^2 = k_o/(UL)$, where k_o is the value of k for $v = U$ in Eq. 95. Since k_o is of the order of $UD/10$, e^2 is of the order of $D/10L$ which is very small. We shall seek solutions by neglecting terms of the order of e^2. Thus the flow in regions away from the zone of dispersion is obtained by neglecting dispersion. Such flows have been studied by others[5]

The horizontal flow is Fig. 13 is to be studied first in order to reveal the effect of gravity on dispersion. Let p(x,y) in the zone of dispersion be

$$p = p_o - \frac{U \mu_o}{\beta} x - \rho_o g y + p' \tag{97}$$

where p_o is the pressure at the origin, while U and μ_o represent the seepage velocity and viscosity respectively of the approaching lighter fluid.

Since the first three terms on the right side of Eq. 97 describe the pressure distribution in the lighter fluid in the absence of dispersion, $p'(x,y)$ is the change of pressure due to dispersion and density difference.

It will be seen that the ratio of the width to the length of the zone of dispersion is of the order of e ($=\sqrt{k_o/(UL)}$, where k_o is the value of k in the approaching lighter fluid). To study the flow in this zone, use the coordinates $\xi = x/L$ and $\eta = y/(eL)$; and the dimensionless variables $\lambda = p'/(e\rho_o gL)$, $\chi = v_x/U$, $\omega = v_y/(eU)$, and $\sigma = c/c_o$. With Eq. 97, Eq. 91 can be written as

$$\chi = \frac{\mu_o}{\mu}(1 - J\frac{\partial\lambda}{\partial\xi}) \tag{98}$$

and

$$\frac{\partial\lambda}{\partial\eta} + \epsilon\sigma = -\frac{e^2\mu\omega}{\mu_o J} = O(e^2) \tag{99}$$

where $\epsilon = \Delta\rho/\rho_o$ and $J = e/I$. Here $I = U\mu_o/(\rho_o g)$ is the hydraulic gradient in the approaching lighter fluid, and is usually much smaller than unity. We will consider cases with J up to the order of unity. With Eq. 90, Eq. 92 becomes

$$(1 + \epsilon\sigma)(\frac{\partial\chi}{\partial\xi} + \frac{\partial\omega}{\partial\eta}) + \epsilon(\chi\frac{\partial\sigma}{\partial\xi} + \omega\frac{\partial\sigma}{\partial\eta}) = 0 \tag{100}$$

In the case of Eq. 94, first consider the term $\nabla\cdot[\rho T \nabla(\frac{c}{\rho})]$. For the two-dimensional flow under consideration,

$$\nabla\cdot[\rho T \nabla(\frac{c}{\rho})] = \frac{\partial}{\partial x}[\rho T_{11}\frac{\partial}{\partial x}(\frac{c}{\rho}) + \rho T_{12}\frac{\partial}{\partial y}(\frac{c}{\rho}) + \frac{\partial}{\partial y}[\rho T_{21}\frac{\partial}{\partial x}(\frac{c}{\rho}) + \rho T_{22}\frac{\partial}{\partial y}(\frac{c}{\rho})] \tag{101}$$

where T_{ij} are the components of T with respect to the x-y axes. To evaluate T_{ij}, let x'-axis lie in the direction of velocity $\vec{v}$ at the point under consideration as shown in Fig. 14. With respect to the x'-y' axes,

$$T'_{mn} = \begin{bmatrix} k' & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix} \tag{102}$$

where k' and k are the coefficients of longi-

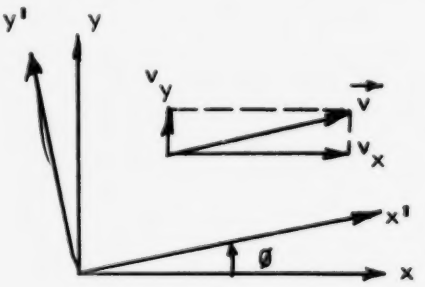

Fig. 14. Transformation of Dispersion Coefficients.

tudinal and lateral dispersion respectively. It is to be shown that many terms in Eq. 101 can be neglected. For estimating order of magnitude, it is sufficient to use b = 1 in Eq. 95, so that k = aDU. Call k' = a'DU. By the transformation rule of tensors,

$$T_{11} = k'\cos^2\phi + k\sin^2\phi = D(a'v_x^2 + av_y^2)/v \tag{103}$$

$$T_{12} = T_{21} = (k' - k)\sin\phi\cos\phi = D(a' - a)v_x v_y/v \tag{104}$$

$$T_{22} = k'\sin^2\phi + k\cos^2\phi = D(a'v_y^2 + av_x^2)/v \tag{105}$$

Eq. 101 can then be written as

$$\begin{aligned}\frac{1}{c_o U}\nabla\cdot\left[\rho\,T\,\nabla\left(\frac{c}{\rho}\right)\right] \doteq e^2\frac{\partial}{\partial\xi}\Bigg[&(1+\epsilon\sigma)\left(\frac{a'}{a}-\frac{\chi^2}{\sqrt{\chi^2+e^2\omega^2}}+\frac{e^2\omega^2}{\sqrt{\chi^2+e^2\omega^2}}\right)\frac{\partial}{\partial\xi}\left(\frac{\sigma}{1+\epsilon\sigma}\right)\\ &+(1+\epsilon\sigma)\left(\frac{a'}{a}-1\right)\frac{\chi\omega}{\sqrt{\chi^2+e^2\omega^2}}\frac{\partial}{\partial\eta}\left(\frac{\sigma}{1+\epsilon\sigma}\right)\Bigg]\\ +\frac{\partial}{\partial\eta}\Bigg[&(1+\epsilon\sigma)\left(\frac{a'}{a}-1\right)\frac{e^2\chi\omega}{\sqrt{\chi^2+e^2\omega^2}}\frac{\partial}{\partial\xi}\left(\frac{\sigma}{1+\epsilon\sigma}\right)\\ &+(1+\epsilon\sigma)\left(\frac{e^2a'\omega^2}{a\sqrt{\chi^2+e^2\omega^2}}+\frac{\chi^2}{\sqrt{\chi^2+e^2\omega^2}}\right)\frac{\partial}{\partial\eta}\left(\frac{\sigma}{1+\epsilon\sigma}\right)\Bigg]\end{aligned} \tag{106}$$

Thus, when terms of the order of e^2 are neglected, it is only necessary to consider dispersion in the y-direction due to T_{22} equal to k computed with $v = v_x$ in Eq. 95.

Eq. 94 can then be written as

$$v_x\frac{\partial c}{\partial x} + v_y\frac{\partial c}{\partial y} = \frac{\rho}{\rho_o}\frac{\partial}{\partial y}\left[\rho k\frac{\partial}{\partial y}\left(\frac{c}{\rho}\right)\right] \tag{107}$$

where, according to Eq. 95,

$$k = k_o\left(\frac{v_x}{U}\right)^b\left(\frac{\mu\rho_o}{\mu_o\rho}\right)^{1-b} = k_o\chi^b\left(\frac{\mu}{\mu_o}\right)^{1-b}\left(\frac{1}{1+\epsilon\sigma}\right)^{1-b} \tag{108}$$

In terms of the dimensionless variables, Eq. 107 becomes

$$\chi \frac{\partial \sigma}{\partial \xi} + \omega \frac{\partial \sigma}{\partial \eta} = (1 + \epsilon\sigma) \frac{\partial}{\partial \eta}\left[(1 + \epsilon\sigma)^{b} \chi^{b} \left(\frac{\mu}{\mu_o}\right)^{1-b} \frac{\partial}{\partial \eta}\left(\frac{\sigma}{1 + \epsilon\sigma}\right)\right] \tag{109}$$

Since μ/μ_o can be expressed as a function of σ, Eqs. 98, 99, 100, and 109 form a set of equations for λ, χ, ω, and σ. Solutions of these equations can be obtained with ϵ as the perturbation quantity if ϵ is sufficiently small.

Let
$$\lambda = \epsilon\lambda_1(\xi,\eta) + \ldots \tag{110}$$

$$\chi = \chi_o(\xi,\eta) + \epsilon\chi_1(\xi,\eta) + \ldots \tag{111}$$

$$\omega = \omega_o(\xi,\eta) + \epsilon\omega_1(\xi,\eta) + \ldots \tag{112}$$

and
$$\sigma = \sigma_o(\xi,\eta) + \epsilon\sigma_1(\xi,\eta) + \ldots \tag{113}$$

Writing

$$\mu = \mu_o + \frac{d\mu}{dc} c + \ldots = \mu_o + \frac{d\mu}{d\rho} \frac{\Delta\rho}{c_o} c + \ldots \tag{114}$$

we have

$$\frac{\mu}{\mu_o} = 1 + \epsilon\gamma\sigma + \ldots \tag{115}$$

where
$$\gamma = \frac{\rho_o}{\mu_o} \left.\frac{d\mu}{d\rho}\right|_{\rho=\rho_o} \tag{116}$$

With Eqs. 111, 112, 113 and 115 in Eqs. 98, 100 and 109, we have to the order of ϵ^0,

$$\chi_o(\xi,\eta) = 1 \tag{117}$$

$$\omega_o(\xi,\eta) = 0 \tag{118}$$

and
$$\frac{\partial \sigma_o}{\partial \xi} = \frac{\partial^2 \sigma_o}{\partial \eta^2} \tag{119}$$

The solution of Eq. 119 for $\sigma_o = 1$ at $\eta = -\infty$, and $\sigma_o = 0$ at $\eta = \infty$ is

$$\sigma_o(\xi,\eta) = \frac{1}{2} \operatorname{erfc}\left(\frac{\eta}{2\sqrt{\xi}}\right) \tag{120}$$

Similarly, we obtain with terms of the order of ϵ from Eqs. 98, 99, 100 and 109,

$$\chi_1 = -\gamma\sigma_o - J\frac{\partial\lambda_1}{\partial\xi} \tag{121}$$

$$\frac{\partial\lambda_1}{\partial\eta} + \sigma_o = 0 \tag{122}$$

$$\frac{\partial\chi_1}{\partial\xi} + \frac{\partial\omega_1}{\partial\eta} + \frac{\partial\sigma_o}{\partial\xi} = 0 \tag{123}$$

$$\frac{\partial\sigma_1}{\partial\xi} + \chi_1\frac{\partial\sigma_o}{\partial\xi} + \omega_1\frac{\partial\sigma_o}{\partial\eta} = \frac{\partial^2\sigma_1}{\partial\eta^2} + \left[b\chi_1 + (1-b)(\gamma-1)\sigma_o\right]\frac{\partial^2\sigma_o}{\partial\eta^2} + b\frac{\partial\sigma_o}{\partial\eta}\frac{\partial\chi_1}{\partial\eta}$$

$$-\left[b(\gamma-1) + 2 - \gamma\right]\left(\frac{\partial\sigma_o}{\partial\eta}\right)^2 \tag{124}$$

With Eq. 120, Eq. 122 yields for the condition that $\lambda_1 = 0$ at $\eta = \infty$

$$\lambda_1(\xi,\eta) = \sqrt{\frac{\xi}{\pi}}\exp\left(-\frac{\eta^2}{4\xi}\right) - \frac{\eta}{2}\operatorname{erfc}\left(\frac{\eta}{2\sqrt{\xi}}\right) \tag{125}$$

From Eq. 121

$$\chi_1(\xi,\eta) = -\frac{\gamma}{2}\operatorname{erfc}\left(\frac{\eta}{2\sqrt{\xi}}\right) - \frac{J}{2\sqrt{\pi\xi}}\exp\left(-\frac{\eta^2}{4\xi}\right) \tag{126}$$

From Eq, 123, with the condition that $\omega_1 = 0$ at $\eta = \infty$,

$$\omega_1(\xi,\eta) = \frac{1}{2\sqrt{\pi\xi}}\left(1 - \gamma - \frac{J\eta}{2\xi}\right)\exp\left(-\frac{\eta^2}{4\xi}\right) \tag{127}$$

With these solutions, Eq. 124 becomes

$$\frac{\partial^2\sigma_1}{\partial\eta^2} - \frac{\partial\sigma_1}{\partial\xi} = \frac{(1-b)+2\gamma b}{4\pi\xi}\exp\left(-\frac{\eta^2}{2\xi}\right) + \frac{(1-b)(1-2\gamma)}{8\sqrt{\pi}}\frac{\eta}{\xi^{3/2}}\operatorname{erfc}\left(\frac{\eta}{2\sqrt{\xi}}\right)\exp\left(-\frac{\eta^2}{4\xi}\right)$$

$$+ \frac{Jb\eta}{4\pi\xi^2}\exp\left(-\frac{\eta^2}{2\xi}\right) \tag{128}$$

In this linear equation, the influence of gravity on the concentration distribution, as represented by the last term involving J, can be studied separately.

This gravitational effect σ^{**} is given by

$$\frac{\partial^2\sigma^{**}}{\partial\eta^2} - \frac{\partial\sigma^{**}}{\partial\xi} = \frac{Jb\eta}{4\pi\xi^2}\exp\left(-\frac{\eta^2}{2\xi}\right) \tag{129}$$

With $\sigma^{**}\sqrt{\xi} = f(\zeta)$ and $\zeta = \eta/\sqrt{\xi}$, this equation is reduced to

$$\frac{d^2f}{d\zeta^2} + \frac{1}{2}\frac{d}{d\zeta}(\zeta f) = \frac{Jb}{4\pi}\zeta\exp\left(-\frac{\zeta^2}{2}\right) \tag{130}$$

For the condition that $f = 0$ at $\zeta = \pm\infty$, the solution can be obtained by repeated integration:

$$\sigma^{**}\sqrt{\xi} = f = -\frac{bJ}{4\sqrt{\pi}}\,\mathrm{erf}\left(\frac{\zeta}{2}\right)\exp\left(-\frac{\zeta^2}{4}\right) \tag{131}$$

The effects of the differences in density and viscosity on σ_1 are shown in Eq. 128 by the factors (1-b) and γ respectively. These effects, as represented by σ^{*}, are given by

$$\frac{\partial^2\sigma^{*}}{\partial\eta^2} - \frac{\partial\sigma^{*}}{\partial\xi} = \frac{(1-b)+2\gamma b}{4\pi\xi}\exp\left(-\frac{\eta^2}{4\xi}\right) + \frac{(1-b)(1-2\gamma)}{8\sqrt{\pi}}\,\frac{\eta}{\xi^{3/2}}\,\mathrm{erfc}\left(\frac{\eta}{2\sqrt{\xi}}\right)\exp\left(-\frac{\eta^2}{4\xi}\right) \tag{132}$$

With $\zeta = \eta/\sqrt{\xi}$, this equation is reduced to

$$\frac{d}{d\zeta}\left(\frac{d\sigma^{*}}{d\zeta}\right) + \frac{\zeta}{2}\frac{d\sigma^{*}}{d\zeta} = \frac{(1-b)+2\gamma b}{4\pi}\exp\left(-\frac{\zeta^2}{2}\right) + \frac{(1-b)(1-2\gamma)}{8\sqrt{\pi}}\,\zeta\,\mathrm{erfc}\left(\frac{\zeta}{2}\right)\exp\left(-\frac{\zeta^2}{4}\right) \tag{133}$$

The solution for the condition that $\sigma^{*} = 0$ at $\zeta = \pm\infty$ is

$$\sigma^{*} = -\frac{(1-b)+2\gamma b}{8}\left[1-\left(\mathrm{erf}\frac{\zeta}{2}\right)^2\right] - \frac{(1-b)(2\gamma-1)}{8\sqrt{\pi}}\left[\frac{2}{\sqrt{\pi}}\exp\left(-\frac{\zeta^2}{2}\right) - \zeta\,\mathrm{erfc}\left(\frac{\zeta}{2}\right)\exp\left(-\frac{\zeta^2}{4}\right)\right] \tag{134}$$

Then

$$\sigma = \sigma_0 + \epsilon\sigma_1 + \ldots = \sigma_0 + \epsilon(\sigma^{**} + \sigma^{*}) + \ldots \tag{135}$$

where σ_0, σ^{**}, and σ^{*} are given by Eqs. 120, 131 and 134 respectively.

It can be seen from Eq. 131 that gravity tends to reduce the width of the zone of dispersion. However, $\epsilon\sigma^{**}$ is proportional to $\epsilon J/\sqrt{\xi}$ or $\epsilon_N\sqrt{k_0}\Big/(I\sqrt{Ux})$ which decreases rapidly along the flow, and is significant only when the hydraulic gradient I is very gentle. Experiments have been performed with the bed of pebbles described in Figs. 2 and 3. The value of β/μ_0 for this bed is experimentally

found to be 0.0066 ft^4/lb-sec with water at 65°F. With water and a dyed salt solution as the fluids in parallel flow ($\epsilon = 0.100$ and $\gamma = +2.0$) at velocities as low as 0.0021 ft/sec, the observed concentration distribution of dye at the 36 outlets (at x = 22 ft) shows no gravitational effect as compared to that observed with water and dyed water ($\epsilon = 0$) at the same velocity. (It is difficult to maintain a steady flow during the 24-hour duration of each test when U is less than 0.0021 ft/sec.) According to Eq. 131 with $\epsilon bJ/\sqrt{\xi} = 0.106$ for the case with U = 0.0021 ft/sec, the maximum magnitude of $\epsilon\sigma^{**}$ is 0.006 at $\zeta = 1.24$ where the observed σ_o (with $\epsilon = 0$) is 0.195±0.018. The change due to $\epsilon\sigma^{**}$ is too small to be detected in these experiments.

When $J/\sqrt{\xi}$ becomes small such that $\sigma^{**} \ll \sigma^{*}$, Eq. 135 becomes

$$\sigma = \sigma_o + \epsilon\sigma^{*} + \ldots \tag{136}$$

For all practical purposes, the effect of $\epsilon\sigma^{*}$ is to shift the concentration distribution $\sigma = \sigma_o$ laterally through a short distance, with $\sigma = \frac{1}{2}$ located at ζ^{*} given by

$$erf(-\frac{\zeta^{*}}{2}) \doteq 2\epsilon\left[\frac{(1-b)+2\gamma b}{8} + \frac{(1-b)(2\gamma-1)}{4\pi}\right] \tag{137}$$

The magnitude of this shift is usually negligible. For example, for fresh water and sea water ($\gamma = +2$ and $\epsilon \doteq 0.03$), this shift in ζ is only 0.19 which is about 5% of the nominal width of the zone of dispersion.

j. Dispersion at Interface of a Two-dimensional Flow

It has been stated in connection with Eq. 96 that, with terms of the order of e^2 neglected, the flow in regions away from the zone of dispersion is obtained by neglecting dispersion. Such outer solutions give the location of the interface and the velocity distribution. In many cases, one of the fluids is at rest[5]. We shall first consider the case with the heavier fluid at rest, as shown in Fig. 15. (Such a case is encountered, for example, in the intrusion of salt-

water into an aquifer[16].) The velocity of the lighter fluid at the interface is given by the outer solution as

$$\bar{u}(x) = U.f(\tfrac{x}{L}) \qquad (138)$$

where L is a representative length of the interface, and U is a representative value of $\bar{u}$. Here x is a boundary-layer coordinate measured along the interface from the first point of contact between the two fluids.

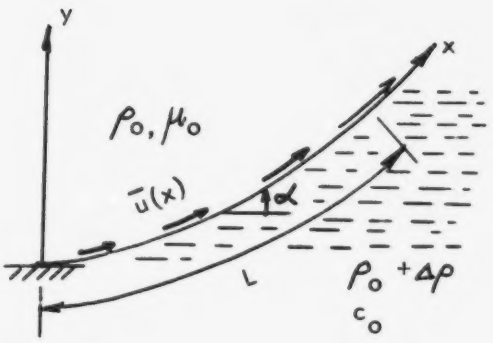

Fig. 15. Flow over a heavier fluid which is at rest.

For the two-dimensional flow shown in Fig. 15, Eq. 91 can be written as

$$v_x = -\frac{\beta}{\mu}\left(\frac{\partial p}{\partial x} + \rho g \sin\alpha\right) \qquad (139)$$

and

$$v_y = -\frac{\beta}{\mu}\left(\frac{\partial p}{\partial y} + \rho g \cos\alpha\right) \qquad (140)$$

where $\alpha(x)$ is the angle between the horizontal and the x-direction. (In using the boundary-layer coordinates, it has been assumed that $\epsilon\frac{d\alpha}{d\xi} \ll 1$.) Since p must be the same in both fluids at the interface, Eq. 139 gives

$$-\frac{U\mu_o}{\beta} f(\tfrac{x}{L}) - \rho_o g \sin\alpha = \left.\frac{\partial p}{\partial x}\right|_{y=0} = -(\rho_o + \Delta\rho) g \sin\alpha \qquad (141)$$

Thus

$$\sin\alpha = \frac{U\mu_o}{\beta \Delta\rho g} f(\tfrac{x}{L}) \qquad (142)$$

and

$$\left.\frac{\partial p}{\partial x}\right|_{y=0} = -\frac{(\rho_o+\Delta\rho)\mu_o U}{\beta \Delta\rho} f(\tfrac{x}{L}) \qquad (143)$$

Call

$$p = p_o - \frac{(\rho_o+\Delta\rho)\mu_o U}{\beta \Delta\rho}\int_0 f(\tfrac{x}{L})\,dx - \rho_o g y \cos\alpha + p' \qquad (144)$$

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16 Henry, H.R., "Salt Intrusion into Fresh-water Aquifers", Journ. of Geophysical Research, Vol. 64, No.11, Nov. 1959.

With Eqs. 142 and 144, Eq. 139 can be written in terms of the dimensionless variables as

$$\chi = -\frac{\mu_0}{\mu}\left[-(1-\sigma).f(\xi) + J\frac{\partial\lambda}{\partial\xi}\right] \tag{145}$$

(In this equation, a term of the order of $\frac{e}{\epsilon}\frac{d\alpha}{d\xi}$ has been neglected.) Eq. 140 can be written as

$$\frac{\partial\lambda}{\partial\eta} + \epsilon\sigma\cos\alpha = -\frac{e^2\mu\omega}{\mu_0^2 J} = O(e^2) \tag{146}$$

Eqs. 100, 109, 145 and 146 form a set of equations for the variables λ, χ, ω and σ.

For cases with small ϵ, solutions can be obtained with ϵ as a perturbation quantity as shown in Eqs. 110 to 113. Solutions will be obtained for $\chi_o(\xi,\eta)$, $\omega_o(\xi,\eta)$ and $\sigma_o(\xi,\eta)$ in these equations. These solutions should be adequate for practical purposes since, as in the flow in Fig. 13, the effects on dispersion due to gravity and differences in density and viscosity, as given by the terms of the order of ϵ, are usually small. Substituting Eqs. 110 to 113 into Eqs. 100, 109 and 145, we have

$$\frac{\partial\chi_o}{\partial\xi} + \frac{\partial\omega_o}{\partial\eta} = 0 \tag{147}$$

$$\chi_o\frac{\partial\sigma_o}{\partial\xi} + \omega_o\frac{\partial\sigma_o}{\partial\eta} = \chi_o^b\frac{\partial^2\sigma_o}{\partial\eta^2} + b\chi^{b-1}\frac{\partial\chi_o}{\partial\eta}\frac{\partial\sigma_o}{\partial\eta} \tag{148}$$

$$\chi_o = (1-\sigma_o)\,f(\xi) \tag{149}$$

Introduce the stream function $\psi_o(\xi,\eta)$, with $\chi_o = \partial\psi_o/\partial\eta$ and $\omega_o = -\partial\psi_o/\partial\xi$; and use ξ and ψ_o, instead of ξ and η, as independent variables. Eq. 148 becomes

$$\frac{1}{\chi_o}\frac{\partial\sigma_o}{\partial\xi} = \chi_o^b\frac{\partial^2\sigma_o}{\partial\psi_o^2} + (1+b)\chi^{b-1}\frac{\partial\chi_o}{\partial\psi_o}\frac{\partial\sigma_o}{\partial\psi_o} \tag{150}$$

This equation can be reduced to an ordinary differential equation by calling

$$X(\xi) = \int_0^\xi\left[f(t)\right]^{1+b}dt \tag{151}$$

$$\tau(\xi,\psi_o) = \frac{\psi_o}{\sqrt{X}}$$

and

$$X_o = f(\xi).F(\tau) \tag{153}$$

Then Eq. 149 gives

$$\sigma_o = 1 - F(\tau) \tag{154}$$

and Eq. 150 becomes

$$\frac{d^2F}{d\tau^2} + \frac{\tau}{2F^{1+b}}\frac{dF}{d\tau} + \frac{1+b}{F}\left(\frac{dF}{d\tau}\right)^2 = 0 \tag{155}$$

This equation has been integrated numerically with a digital computer to obtain $F(\tau)$ for $b = \frac{3}{4}$. In each step of the integration, three computed values at $\Delta\tau = 0.001$ apart are used to compute the next value. The boundary conditions $F(-\infty)= 0$ and $F(\infty)=1$ are met by using trial values of F and $\frac{dF}{d\tau}$ at $\tau = -1$. The relationship between τ and η can be obtained by integrating $d\eta = d\psi_o/\chi_o$ at constant ξ and X:

$$\int_{\eta^*} d\eta = \frac{\sqrt{X(\xi)}}{f(\xi)}\int_{\tau^*} \frac{d\tau}{F(\tau)} \tag{156}$$

where η^* and τ^* are values of η and τ respectively at $F = \frac{1}{2}$. This integration has been performed numerically with a digital computer. With this relationship between η and τ, and that between F and η, $(\eta - \eta^*)$ can be related to F as shown in Fig. 16. The value of η^* for $F = \frac{1}{2}$ cannot be determined in this approximate analysis[17]. According to available experimental data[18], it seems that one can use $\eta^* = 0$.

To illustrate the dispersion described by this solution, consider the case in which the velocity at the interface is given by $f(\xi)= \xi^m$ in Eq. 138 $(m>\frac{-1}{1+b})$. In this case,

$$\frac{f(\xi)\eta}{\sqrt{X(\xi)}} = \sqrt{1+(1+b)m}\;\xi^{(1-b)m/2}\frac{\eta}{\sqrt{\xi}}$$

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17 Crane, L.J., "The Laminar and Turbulent Mixing of Jets of Compressible Fluids, Part II", Journ. Fluid Mech., Vol. 3, 81, 1957.

18 Rumer, R.R. and Harleman, D.R.F., "Intruded Salt-Water Wedge in Porous Media", Proc. Amer. Soc. Civ. Engrs., Vol. 89, No.HY6, Nov. 1963.

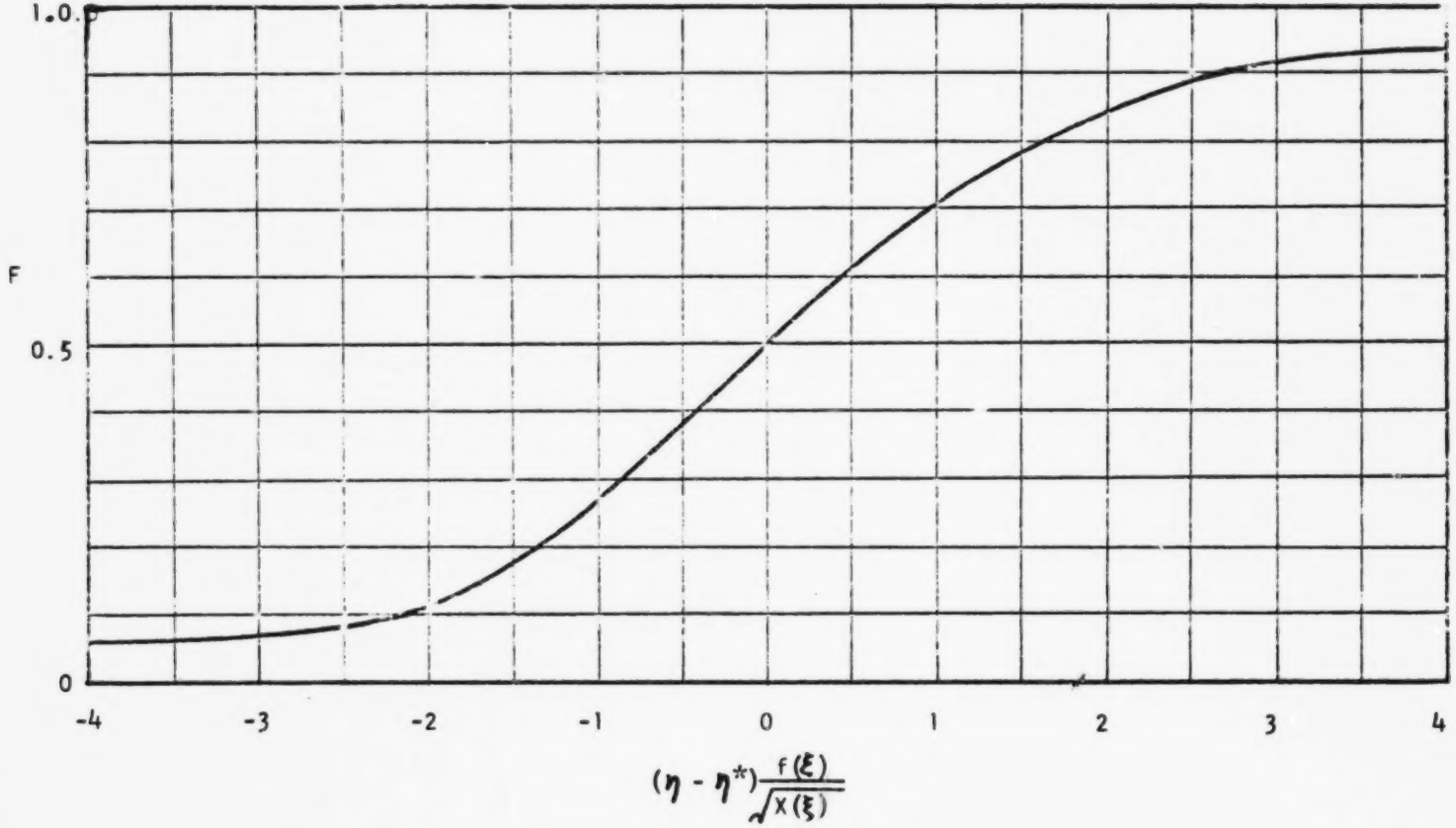

Fig. 16. Solution of Eq. 155 for $b = \frac{3}{4}$.

For the case with m = 0, σ_o = 0.1 and 0.9 occur at $\frac{\eta}{\sqrt{\xi}}$ = 2.70 and -2.14 respectively. The zone of dispersion is thus narrower on the side of the fluid at rest. (As a comparison, we have, from Eq. 120 for the case of two fluids in parallel flow, σ_o = 0.1 and 0.9 at $\frac{\eta}{\sqrt{\xi}}$ = 1.81 and -1.81 respectively.) In cases with m>0, the zone of dispersion is narrower (except near the origin) since the streamline of the moving liquid converge.

In the case where the lighter fluid is at rest as shown in Fig. 17, with the velocity of the heavier fluid at the interface given by Eq. 138, a similar analysis shows that

$$\chi_o = f(\xi).F \tag{157}$$

but

$$\sigma_o = F \tag{158}$$

where F is the same function as shown in Fig. 16.

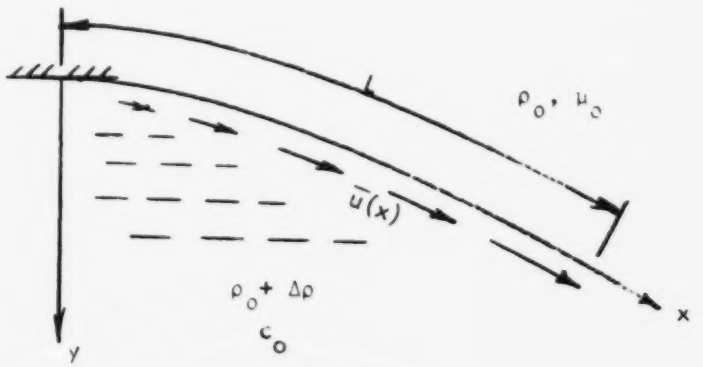

Fig. 17. Flow under a lighter fluid which is at rest.

From the solutions presented above, a qualitative description of the concentration distribution can be obtained for cases where both liquids are flowing along the interface. Take for example σ_o in Eq. 154 for the flow over a resting heavier liquid and σ_o in Eq. 120 for the parallel flow. In the latter flow, the two liquids have, to the order of e^o, the same constant velocity. Thus we have for this flow $f = 1$, $X=\xi$ and $\eta^* = 0$. Equation 120 can therefore be written as

$$\sigma_o = \frac{1}{2} \operatorname{erfc} \frac{(\eta-\eta^*)f}{2\sqrt{X}}$$

This equation and Eq. 154 have been plotted in Fig. 18. In spite of the fact that the heavier liquid is at rest in one flow, and is moving at practically the same velocity as the lighter liquid in the other, the two curves are quite similar. It may be surmised that, when the heavier liquid is moving with a velocity lower than that of the lighter liquid, the concentration distribution will fall between the two curves in Fig. 18. Similarly, Fig. 19 has been plotted with Eq. 158 for cases with the lighter liquid moving with a lower velocity. As in Fig. 17, η is positive on the side of the heavier liquid in Fig. 19. The zone of dispersion is wider when one of the liquids is at rest, due to the fact that, since $b<1$, a lower velocity in the zone gives a wider zone.

k. Dispersion at Interface of a Three-dimensional Flow

In the previous section, solutions have been obtained, by neglecting terms of the order of e, for dispersion at the interface of two-dimensional flows. It is to be shown in this section that these solutions are also applicable to the similar problem in three-dimensional flows.

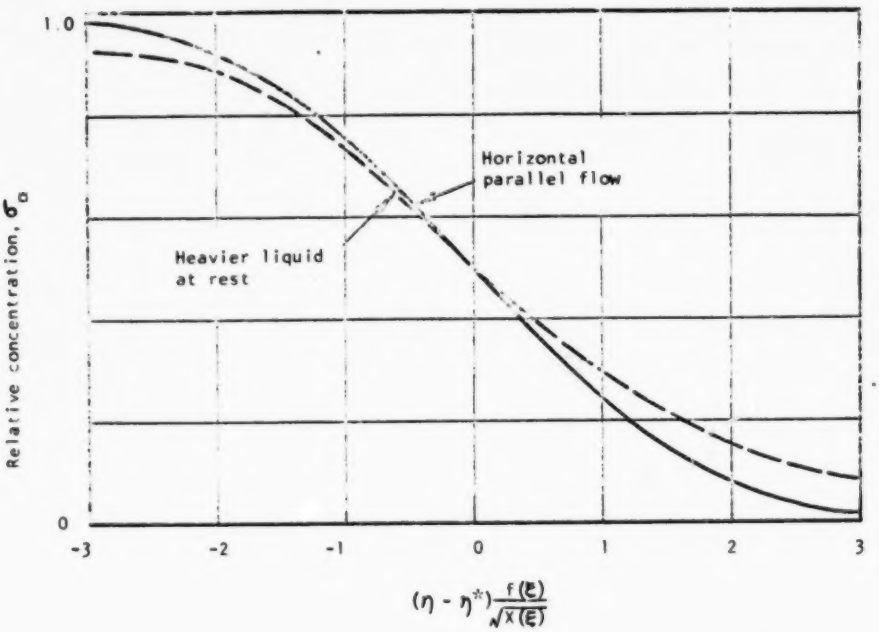

Fig. 18. Concentration distribution when heavier liquid moves more slowly

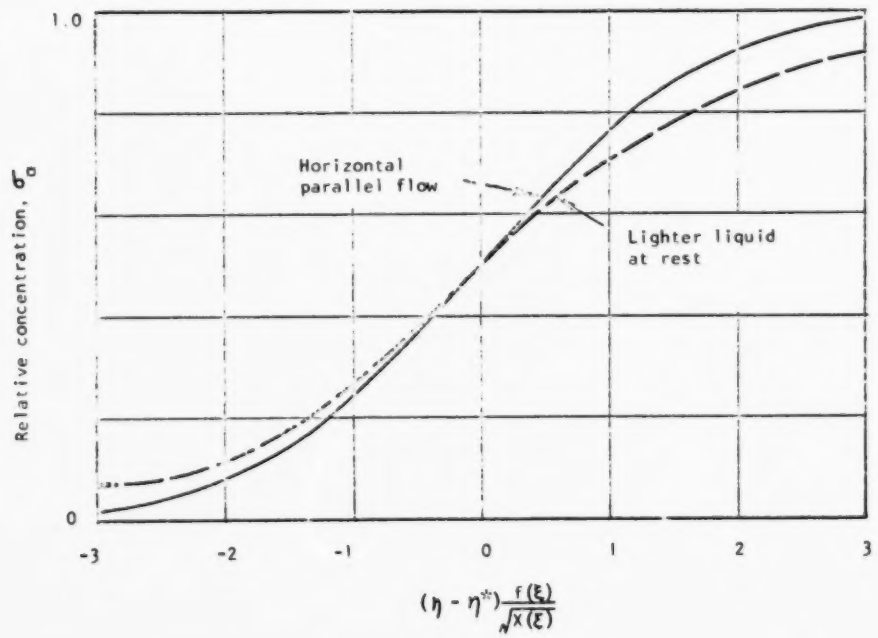

Fig. 19. Concentration distribution when lighter liquid moves more slowly.

Consider the case with the heavier fluid at rest. For a given point 0 on the interface, let the x-direction lie along the streamline given by the outer solution, and the y-direction be normal to the interface and directed towards the moving fluid, as shown in Fig. 20. The outer solution also furnishes the velocity of the moving fluid at the interface

$$\bar{u}(x,z) = U.f(x,z) \qquad (159)$$

and the pressure $\bar{p}(x,z)$ there. From Eq. 91, we have for the neighborhood of 0

$$v_x = -\frac{\beta}{\mu}\left(\frac{\partial p}{\partial x} + \rho g \cos\theta_1\right) \qquad (160)$$

$$v_y = -\frac{\beta}{\mu}\left(\frac{\partial p}{\partial y} + \rho g \cos\theta_2\right) \qquad (161)$$

$$v_z = -\frac{\beta}{\mu}\left(\frac{\partial p}{\partial z} + \rho g \cos\theta_3\right) \qquad (162)$$

where the angles θ_1, etc. are measured from the vertical as shown in Fig. 20.

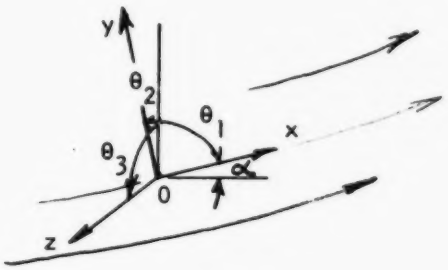

Fig. 20. Three-dimensional flow over a heavier fluid which is at rest.

Since $v_z = 0$ at the interface for both fluids, the last equation gives

$$\frac{\partial \bar{p}}{\partial z} = -\rho_o g \cos\theta_3 = -(\rho_o + \Delta\rho) g \cos\theta_3 \qquad (163)$$

Thus $\cos\theta_3 = 0$ and $\frac{\partial \bar{p}}{\partial z} = 0$. In other words, $\bar{p}$ only varies along a streamline, and the x-y axes form a vertical plane with $\theta_2 = \alpha$ and $\cos\theta_1 = \sin\alpha$, α being the angle from the horizontal to the x-direction.

Eqs. 160 and 161 now become identical to Eqs. 139 and 140 for two-dimensional flows. From Eq. 139, we obtain the relationship among $\frac{d\bar{p}}{dx}$, the angle α, and the velocity $\bar{u}$ in Eqs. 142 and 143. Since $\bar{p}$ is independent of z, these equations show that α and $\bar{u}$ vary only along the streamlines. The definition of p' in Eq. 144 can again be used, and Eqs. 160 and 161 are reduced to Eqs. 145 and 146 as in the case of two-dimensional flows.

The continuity equation in Eq. 92 is now

$$\rho\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) + \left(v_x \frac{\partial \rho}{\partial x} + v_y \frac{\partial \rho}{\partial y} + v_z \frac{\partial \rho}{\partial z}\right) = 0 \qquad (164)$$

While v_x is of the order of U, v_y and v_z caused by dispersion are of the order of eU. If the order of z is L, as is x, this equation can be rewritten as

$$(1 + \epsilon\sigma)\left(\frac{\partial \chi}{\partial \xi} + \frac{\partial \omega}{\partial \eta}\right) + \epsilon\left(\chi \frac{\partial \sigma}{\partial \xi} + \omega \frac{\partial \sigma}{\partial \eta}\right) = 0(e) \qquad (165)$$

Neglecting terms of the order of e, we have this equation reduced to Eq. 100 for two-dimensional flows. Similarly, the dispersion in the z-direction can be shown to be negligible, and Eq. 109 for two-dimensional flows is also applicable to the problem under consideration.

Now that the problem is described by Eqs. 100, 109, 145 and 146, as in the case of two-dimensional flows, and the boundary conditions are identical, the solutions obtained in the previous section are therefore applicable to three-dimensional flows.

l. Appendix - Notation

The following symbols have been adopted for use in this report:

a_1, a_2 = coefficients in Eqs. 6 and 7 respectively;

a, b = coefficients and exponent in Eq. 16 (except in Eq. 9);

c = contaminant concentration in mass per unit volume of soil;

c_o = value of c in lower fluid in Figs. 3, 13, 15, 17 & 20;

d_1, d_2 = linear measurements describing soil grains;

D = 50% grain size of soil;

e = $\sqrt{k_o/(UL)}$;

$f(\xi)$ = function describing $\bar{u}$ in Sections i and j;

F = function describing χ_o in Eq. 153;

$\vec{g}$ = gravitational acceleration;

I = $\mu_o U/(\beta \rho_o g)$;

J = e/I;

k = coefficient of transverse dispersion;

k' = coefficient of longitudinal dispersion;

k_o = value of k for seepage velocity U;

T_{ij}, T'_{mn} = components of T with respect to x-y and x'-y' axes respectively;

K = value of k on s-axis;

$\overline{(\frac{K}{V})}$ = mean value of $\frac{K}{V}$ between source and section (see Eq. 23);

L = radius of curvature of s-axis in Fig. 9, or representative length of interface in Figs. 13, 15 & 17;

m, n = coordinates normal to s-axis (see Figs. 7 and 12);

M = strength of point source in mass per unit time;

N = strength of line source in mass per unit length per unit time;

p = pressure;

p_o = pressure at origin;

p' = change of pressure due to contamination;

P = porosity of soil;

Q = discharge per unit length of well;

r = radial coordinate;

R, R' = rates of fractional lateral variation of v (see Eqs. 20 and 66);

s = coordinate along streamline of contamination (see Figs. 7 and 12);

t = time;

T = tensor coefficient of dispersion;

$\bar{u}$ = seepage velocity at an interface;

U = approaching seepage velocity in Figs. 10a and 13 (except in Eq. 9), or representative velocity at an interface in Fig. 13, 15 & 17.

$\vec{v}$ = seepage velocity, discharge per unit area of void;

v_n, v_s, v_x, v_y, v_z = components of $\vec{v}$ in the n-, s-, x-, y- and z- directions respectively;

V = value of v on s-axis;

w = half-width of trail of contaminant;

x,y,z = coordinates;

X = Function defined in Eq. 151

α = angle of inclination of interface (See Figs. 15 & 20);

β = intrinsic permeability as defined in Eq. 91;

γ = dimensionless material constant defined in Eq. 116;

Δ, δ = shift of trail (see Figs. 7 and 12, and Eqs. 25 and 69);

ϵ = $\Delta\rho/\rho_o$;

ζ = coordinate measured from line of maximum cv in Section g (See Fig. 12), or $\eta/\sqrt{\xi}$ in Sections i and j;

ζ^* = shift in $\eta/\sqrt{\xi}$ of concentration distribution due to $\varepsilon\sigma^*$;

η = $\overline{(\frac{K}{V})}$ s in Sections d to g, or $\frac{y}{eL}$ in Sections i to k;

η^* = value of η for $F = \frac{1}{2}$;

θ = cylindrical coordinates (See Fig. 9);

$\theta_1,\theta_2,\theta_3$ = angles between coordinate axes with the vertical in Fig. 20;

λ = $p'/(e\rho_o gL)$;

λ_1 = function defined in Eq. 110;

μ = absolute viscosity;

μ_o = absolute viscosity of upper fluid in Figs. 13, 15, 17, and 20;

ν = kinematic viscosity;

ξ = coordinate measured from line of maximum cv in Sections d to g, or x/L in Sections i to k;

$\Delta\rho$ = density difference between the two fluids in Figs. 13, 15, 17, and 20;

ρ = density of fluid;

ρ_o = density of upper fluid in Figs. 13, 15, 17, and 20;

σ = c/c_o;

σ_o, σ_1 = functions defined in Eq. 113;

σ^{**} = part of σ_1 due to gravity;

σ^{*} = part of σ_1 due to density & viscosity differences;

τ = $\psi_o / \sqrt{X}$;

τ^{*} = value of τ at $F = \frac{1}{2}$;

X = v_x/U;

X_o, X_1 = functions defined in Eq. 111;

ψ_o = dimensionless Lagrange stream function;

ψ = Lagrange stream function;

Ψ = Stokes stream function;

ω = v_y/Ue;

ω_o, ω_1 = functions defined in Eq. 112.

(2) List of Publications

Li, W.H., "Band Diffusion with Variable Parameters along Flow", Proc. Amer. Soc. Civ. Engrs., Vol. 90 No. EM4, Oct. 1964.

Li, W.H., "Discussion of Band Diffusion", Proc. Amer. Soc. Civ. Engrs., Vol. 91, No. EM6, Dec. 1965.

Li, W.H. and Lai, F.H., "Experiments on Lateral Dispersion in Porous Media", Proc. Amer. Soc. Civ. Engrs., Vol. 92, No. HY6, Nov. 1966.

Li, W.H. and Canale, R.P., "Dispersion from Sources in Non-uniform Seepage Flow", Proc. Amer. Soc. Civ. Engrs., Vol. 93, No. HY3, May 1967.

Li, W.H. and Yeh, G.T., "Dispersion at the Interface of Miscible Liquids in a Soil" scheduled to appear shortly in Water Resources Research, American Geophysical Union.

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